04 - kinematic equations - kinematics of growth

04 - kinematic equations

day	date		topic
tue	jan	10	motivation - everything grows!
thu	jan	12	basics maths - notation and tensors
tue	jan	17	basic kinematics - large deformation and growth
thu	jan	19	kinematics - growing hearts
tue	jan	24	guest lecture - growing skin
thu	jan	26	guest lecture - growing leaflets
tue	jan	31	basic balance equations - closed and open systems
thu	feb	02	basic constitutive equations - growing tumors
tue	feb	07	volume growth - finite elements for growth
thu	feb	09	volume growth - growing arteries
tue	feb	14	volume growth - growing skin
thu	feb	16	volume growth - growing hearts
tue	feb	21	basic constitutive equations - growing bones
thu	feb	23	density growth - finite elements for growth
tue	feb	28	density growth - growing bones
thu	mar	01	everything grows! - midterm summary
tue	mar	06	midterm
thu	mar	08	remodeling - remodeling arteries and tendons
tue	mar	13	class project - discussion, presentation, evaluation
thu	mar	15	class project - discussion, presentation, evaluation
thu	mar	15	written part of final projects due

where are we???

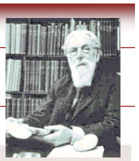
growth, remodeling and morphogenesis

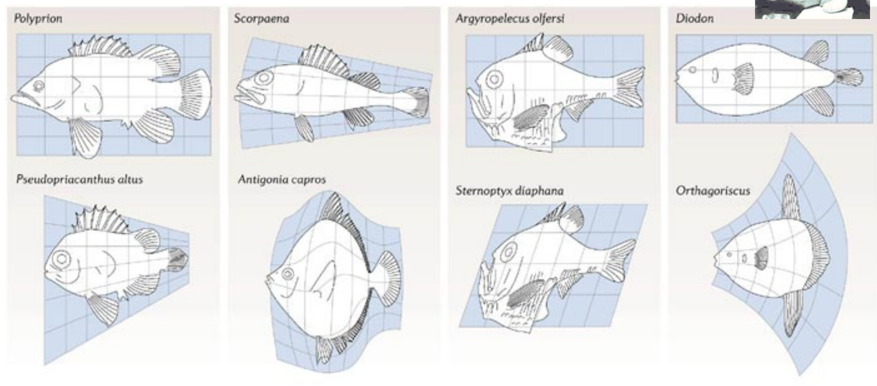
growth [grouθ] which is defined as added mass, can occur through cell division (hyperplasia), cell enlargement (hypertrophy), secretion of extracellular matrix, or accretion @ external or internal surfaces. negative growth (atrophy) can occur through cell death, cell shrinkage, or resorption. in most cases, hyperplasia and hypertrophy are mutually exclusive processes. depending on the age of the organism and the type of tissue, one of these two growth processes dominates.

taber „biomechanics of growth, remodeling and morphogenesis" [1995]

introduction

scaling growth

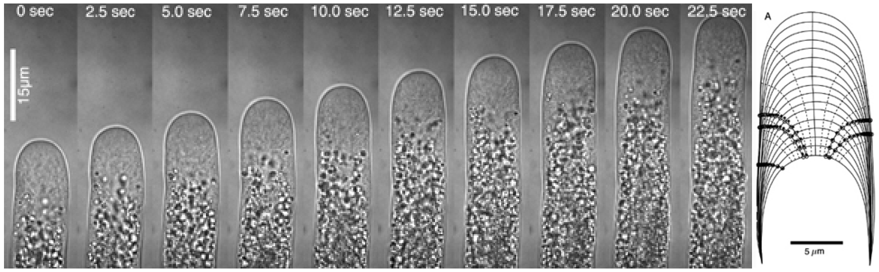




sir d'arcy thompson "on growth and form" [1917]

kinematics of growth

tip growth



time lapse sequence of a growing lily pollen tube. note that the morphology of the tube is drawn by the expanding tip and does not change behind it. tip growth is a common mode of cell morphogenesis observed in root hairs, fungal hyphae, pollen tubes, and many unicellular algae. these organisms have cell walls with distinct polymer compositions and structures.

dumais, long, shaw (2004)

surface growth

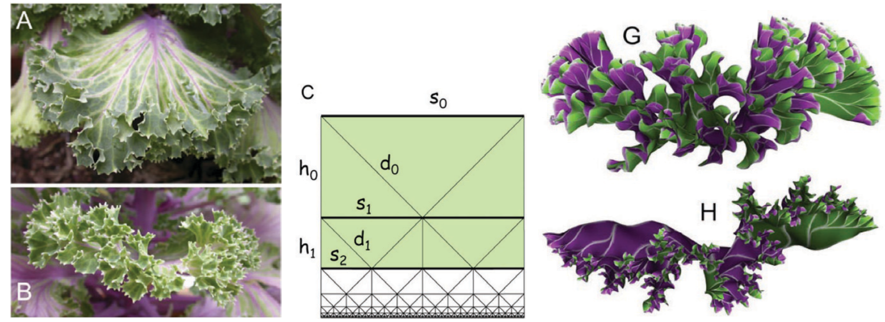
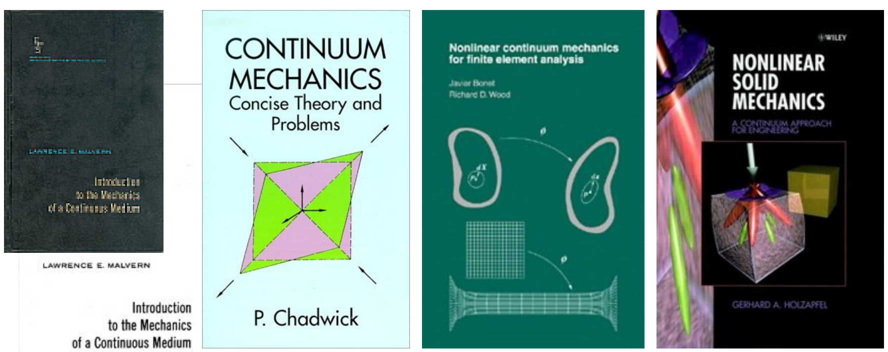


Fig. 5. Simulation study of surfaces with a fractal cascade of waves at the margin. (A) A kale (a variety of *Brassica oleracea*) leaf showing a superposition of waves with a decreasing amplitude and wavelength towards the leaf margin. (B) The fractal character of the leaf margin. (C) A computational representation of a leaf. The surface is divided into rows of geometrically similar rectangles, each row with twice the number of rectangles as its predecessor. The first two rows are highlighted in green. Each rectangle is further subdivided into three triangles. Proportions are controlled by the scaling ratio r, initially set to $\frac{1}{2}$, such that $s_{i+1}=h_i=rs_i$ and $d_i=r\sqrt{2}s_i$ for i=0,1,2,....

prusinkiewicz & de reuille "constraints of space in plant development" [2010]

suggested reading



malvern le: introduction to the mechanics of a continuous medium, prentice hall, 1969
chadwick p: continuum mechanics - concise theory and problems, dover reprint, 1976
bonet j, wood rd: nonlinear continuum mechanics for fe analysis, cambridge university press, 1997
holzapfel ga: nonlinear solid mechanics, a continuum approach for engineering, john wiley & sons, 2000

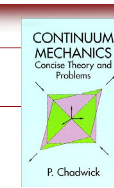
the potato  equations

- kinematic equations - what's strain? $\epsilon = \frac{\Delta l}{l}$
  general equations that characterize the deformation of a physical body without studying its physical cause
- balance equations - what's stress? $\sigma = \frac{F}{A}$
  general equations that characterize the cause of motion of any body
- constitutive equations - how are they related? $\sigma = E\,\epsilon$
  material specific equations that complement the set of governing equations

the potato  equations

- kinematic equations - why not $\epsilon = \frac{\Delta l}{l}$?
  - inhomogeneous deformation » non-constant
  - finite deformation » non-linear $\quad \boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  - inelastic deformation » growth tensor $\quad \boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$
- balance equations - why not $\sigma = \frac{F}{A}$? $\quad \mathrm{Div}(\boldsymbol{P}) + \rho\, \boldsymbol{b}_0 = \boldsymbol{0}$
  - equilibrium in deformed configuration » multiple stress measures
- constitutive equations - why not $\sigma = E\, \epsilon$?
  - finite deformation » non-linear $\quad \boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  - inelastic deformation » internal variables $\quad \boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{g}})$

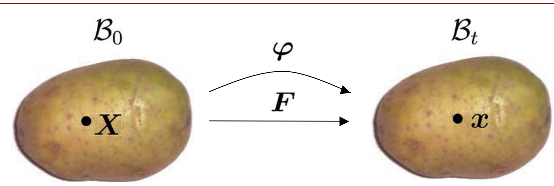
kinematic equations



kinematics [kɪnəˈmætɪks] is the study of motion per se, regardless of the forces causing it. the primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.

Chadwick „Continuum mechanics" [1976]

potato - kinematics



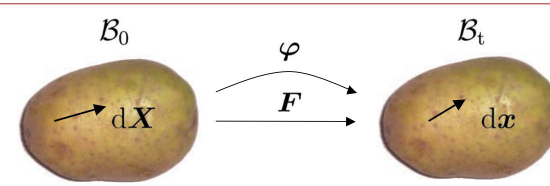
- nonlinear deformation map $\boldsymbol{\varphi}$

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) \quad \text{with} \quad \boldsymbol{\varphi} : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$$

- spatial derivative of $\boldsymbol{\varphi}$ - deformation gradient

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{F} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad \boldsymbol{F} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$$

potato - kinematics



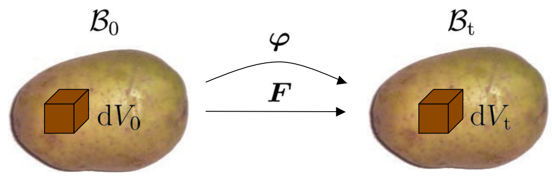
- transformation of line elements - deformation gradient F_{ij}

$$\mathrm{d}x_i = F_{ij}\, \mathrm{d}X_j \quad \text{with} \quad F_{ij} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad F_{ij} = \left.\frac{\partial \varphi_i}{\partial X_j}\right|_{t \text{ fixed}}$$

- uniaxial tension (incompressible), simple shear, rotation

$$F_{ij}^{\mathrm{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F_{ij}^{\mathrm{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{ij}^{\mathrm{rot}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

potato - kinematics



- transformation of volume elements - determinant of $\boldsymbol{F}$

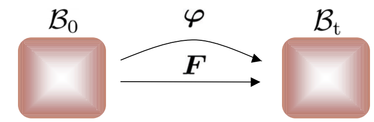
$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3] = \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

$$\mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3] = \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3]) = \det(\boldsymbol{F})\,\det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

- changes in volume - determinant of deformation tensor J

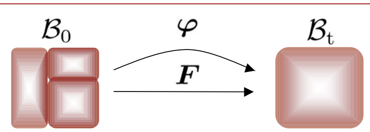
$$\mathrm{d}V_t = J\,\mathrm{d}V_0 \qquad J = \det(\boldsymbol{F})$$

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

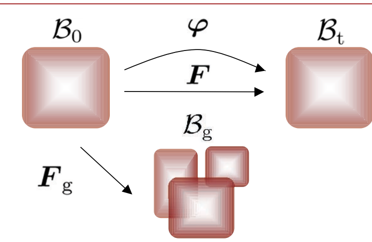
kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

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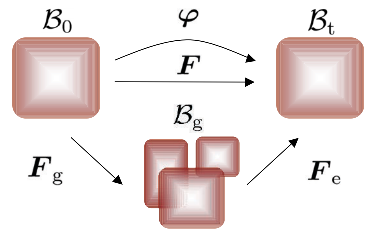
[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

kinematics of finite growth



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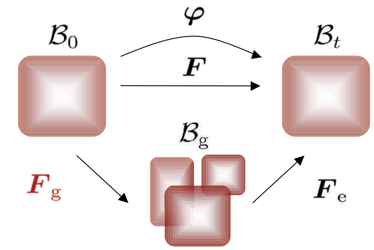
[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$

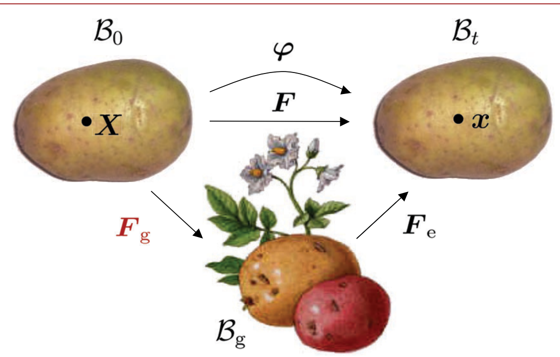
growth tensor



multiplicative decomposition

Lee [1969], Simo [1992], Rodriguez, Hoger & Mc Culloch [1994], Epstein & Maugin [2000], Humphrey [2002], Ambrosi & Mollica [2002], Himpel, Kuhl, Menzel & Steinmann [2005]

potato - kinematics of finite growth



- incompatible growth configuration $\mathcal{B}_g$ & growth tensor $\boldsymbol{F}_g$

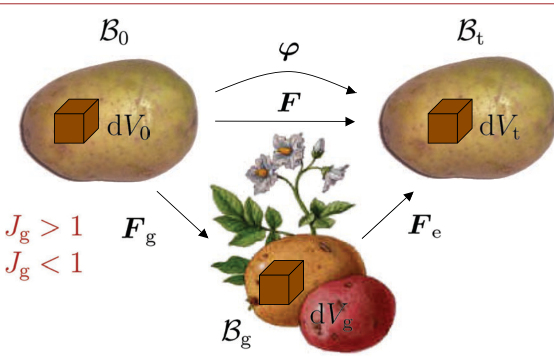
$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$

rodriguez, hoger & mc culloch [1994]

potato - kinematics of finite growth

growth $J_g > 1$

resorption $J_g < 1$

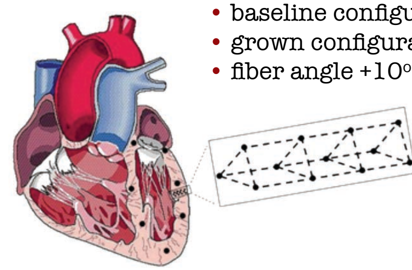


- changes in volume - determinant of growth tensor J_g

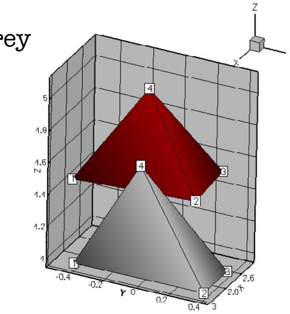
$$\mathrm{d}V_g = J_g \, \mathrm{d}V_0 \qquad J_g = \det(\boldsymbol{F}_g)$$

me338a - continuum mechanics - 2010

kinematics of cardiac growth



given: coordinates of
- baseline configuration X, grey
- grown configuration x, red
- fiber angle +10°



X1 = [+2.80; -0.27; +3.75];
X2 = [+2.80; +0.53; +3.75];
X3 = [+2.50; +0.52; +3.75];
X4 = [+2.80; +0.13; +4.45];

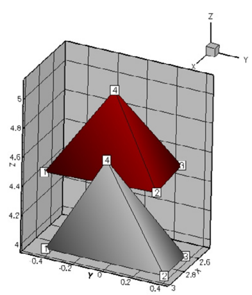
x1 = [+2.62; -0.44; +4.34];
x2 = [+2.62; 0.44; +4.34];
x3 = [+2.32; 0.44; +4.35];
x4 = [+2.60; 0.07; +4.85];

kinematics of cardiac growth

[1] Determine three vectors $\mathrm{d}\boldsymbol{X}_i$ that span the tetrahedron at baseline.
Take an arbitrary point of the tetrahedron as origin, e.g., $\boldsymbol{X}_4$, and calculate the three vectors $\mathrm{d}\boldsymbol{X}_1$, $\mathrm{d}\boldsymbol{X}_2$, and $\mathrm{d}\boldsymbol{X}_3$ from the origin to any other point using the coordinates $\boldsymbol{X}$ at baseline such that $\mathrm{d}\boldsymbol{X}_i = \boldsymbol{X}_i - \boldsymbol{X}_4$ for $i = 1, 2, 3$.

```
matlab
dX1 = X1 - X4
dX2 = X2 - X4
dX3 = X3 - X4
```

dX1 = [+0.00, -0.40, -0.70]
dX2 = [+0.00, +0.40, -0.70]
dX3 = [-0.30, +0.39, -0.70]

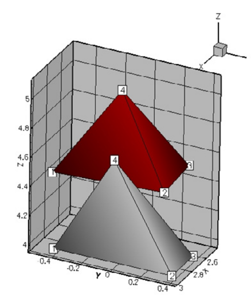


kinematics of cardiac growth

[2] Determine the same three vectors $\mathrm{d}\boldsymbol{x}_i$ that span the tetrahedron after growth.
Take the same point as origin, e.g., $\boldsymbol{x}_4$, and calculate the vectors $\mathrm{d}\boldsymbol{x}_1$, $\mathrm{d}\boldsymbol{x}_2$, and $\mathrm{d}\boldsymbol{x}_3$ from the origin to any other point using the coordinates $\boldsymbol{x}$ after growth such that $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{x}_i - \boldsymbol{x}_4$ for $i = 1, 2, 3$.

```
matlab
dx1 = x1 - x4
dx2 = x2 - x4
dx3 = x3 - x4
```

dx1 = [+0.02, -0.51, -0.51]
dx2 = [+0.02, +0.37, -0.51]
dx3 = [-0.28, +0.37, -0.50]



[3] Determine the growth tensor $\boldsymbol{F}^g$ that maps the baseline line elements $\mathrm{d}\boldsymbol{X}_i$ onto the grown line elements $\mathrm{d}\boldsymbol{x}_i$.
The growth tensor maps line elements according to $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{F}^g \cdot \mathrm{d}\boldsymbol{X}_i$. The application of this mapping to all three line elements $\mathrm{d}\boldsymbol{X}_i$ defines three vector valued equations, i.e., nine equations to solve for the nine components of $\boldsymbol{F}^g$. To obtain a more compact notation, rearrange all baseline line elements from [1] and all grown line elements from [2] in 3×3 matrices, i.e., $\boldsymbol{C} := [\ \mathrm{d}\boldsymbol{X}_1;\ \mathrm{d}\boldsymbol{X}_2;\ \mathrm{d}\boldsymbol{X}_3\ ]$ and $\boldsymbol{c} := [\ \mathrm{d}\boldsymbol{x}_1;\ \mathrm{d}\boldsymbol{x}_2;\ \mathrm{d}\boldsymbol{x}_3\ ]$. Now, determine the growth tensor $\boldsymbol{F}^g$ by using the equation $\boldsymbol{F}^g \cdot \boldsymbol{C} = \boldsymbol{c}$, thus $\boldsymbol{F}^g = \boldsymbol{c} \cdot \boldsymbol{C}^{-1}$.

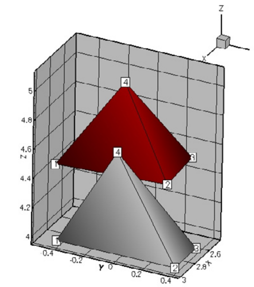
```
matlab
C = [ dX1 dX2 dX3 ];        c = [ dx1 dx2 dx3 ];        F = c/C;
dx1_check = F * dX1;    dx2_check = F * dX2;    dx3_check = F * dX3;
```

$$F = \begin{matrix} +1.0000 & 0.0000 & -0.0286 \\ -0.0367 & +1.1000 & +0.1000 \\ -0.0333 & 0.0000 & +0.7286 \end{matrix}$$

[4] Control your results by calculating $\mathrm{d}\boldsymbol{x}_i = \boldsymbol{F}^g \cdot \mathrm{d}\boldsymbol{X}_i$.
Do the calculated grown line elements $\mathrm{d}\boldsymbol{x}_i$ match the ones you had calculated in [2]?

```
matlab
dx1_check = F * dX1;
dx2_check = F * dX2;
dx3_check = F * dX3;
```

```
dx1_check = [+0.02, -0.51, -0.51]
dx2_check = [+0.02, +0.37, -0.51]
dx3_check = [ -0.28, +0.37, -0.50]
```



[5] Determine the grown fiber direction $\boldsymbol{n}^{\text{fib}} = \boldsymbol{F}^g \cdot \boldsymbol{N}^{\text{fib}}$.
The growth tensor can be used to map the measured baseline fiber direction $\boldsymbol{N}^{\text{fib}}$ onto the grown fiber direction $\boldsymbol{n}^{\text{fib}}$. Determine $\boldsymbol{n}^{\text{fib}}$ and comment on how $\boldsymbol{N}^{\text{fib}}$ and $\boldsymbol{n}^{\text{fib}}$ deviate.

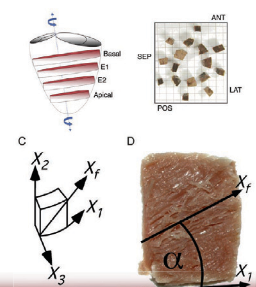
```
matlab
alpha = 10.0;
N_fib = [0.0; -cosd(alpha); sind(alpha)]
n_fib = F * N_fib;
theta = acosd((n_fib'*N_fib)/(norm(n_fib)*norm(N_fib)))
```

```
N_fib = [ 0.0000, -0.9848, +0.1736]
n_fib  = [-0.0050, -1.0659, +0.1265]
theta = 3.2420
```


[6] Determine the fiber stretch upon growth $\lambda^g = \sqrt{\boldsymbol{n}^{\text{fib}} \cdot \boldsymbol{n}^{\text{fib}}}$.
Since the fiber orientation $\boldsymbol{N}^{\text{fib}}$ was given as a unit vector, the length of the grown vector $\boldsymbol{n}^{\text{fib}} = \boldsymbol{F} \cdot \boldsymbol{N}^{\text{fib}}$ corresponds to the relative change in fiber length, i.e., the amount of growth along the fiber direction, $\lambda^g = \sqrt{\boldsymbol{n}^{\text{fib}} \cdot \boldsymbol{n}^{\text{fib}}} = \sqrt{\boldsymbol{N}^{\text{fib}} \cdot \boldsymbol{F}^{g\,t} \cdot \boldsymbol{F}^g \cdot \boldsymbol{N}^{\text{fib}}}$.

```
matlab
lambda = sqrt(n_fib'*n_fib)
```

```
lambda = 1.0734
```



ennis, nguyen, riboh, wigström, harri
ton, daughters, ingels, miller [2010]

kinematics of cardiac growth

[7] Determine the second order Green Lagrange strain tensor $\boldsymbol{E} = \frac{1}{2}\,[\,\boldsymbol{F}^{t} \cdot \boldsymbol{F} - \boldsymbol{I}\,]$.
$\boldsymbol{E}$ is called the Green Lagrange strain tensor and it is used to characterize strains with respect to the reference configuration in a finite strain setting.

[8] Determine the displacement gradient tensor $\boldsymbol{H} = \boldsymbol{F} - \boldsymbol{I}$.
$\boldsymbol{H} = \nabla \boldsymbol{u}$ is the nonsymmetric displacement gradient tensor which can also be expressed as $\boldsymbol{H} = \partial \boldsymbol{u} \,/\, \partial \boldsymbol{X} = \partial [\boldsymbol{x} - \boldsymbol{X}] \,/\, \partial \boldsymbol{X} = \boldsymbol{F} - \boldsymbol{I}$.

```
matlab
E = 1/2 * (F'*F - eye(3))
```

```
matlab
H = F - eye(3)
```

```
    0.0012  -0.0202  -0.0283
E= -0.0202   0.1050   0.0550
   -0.0283   0.0550  -0.2292
```

```
    0.0000   0        -0.0286
H= -0.0367   0.1000    0.1000
   -0.0333   0        -0.2714
```


kinematics of cardiac growth

[9] Determine the small strain tensor $\boldsymbol{\epsilon} = \frac{1}{2}\,(\boldsymbol{H} + \boldsymbol{H}^{t})$.
Compare the small strain approximation $\boldsymbol{\epsilon}$ with the large strain Green Lagrange tensor $\boldsymbol{E}$ and comment on your results.

[10] Determine the normal strain $\epsilon_n = \boldsymbol{N}^{\text{fib}} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{N}^{\text{fib}}$.
Compare the small strain approximation of the normal strain ϵ_n with the large strain fiber stretch λ^{g}.

```
matlab
epsilon = 1/2 * (H+H')    eps_n = N_fib'*epsilon*N_fib;
```

```
           0.0000  -0.0183  -0.0310
epsilon = -0.0183   0.1000   0.0500     eps_n = 0.0717
          -0.0310   0.0500  -0.2714
```

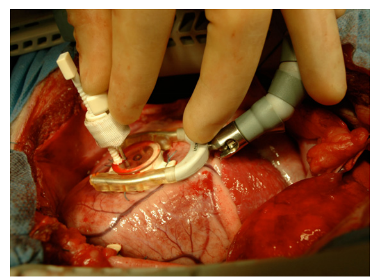

kinematics of cardiac growth

[11] Determine the volume change $J^{g} = \det(\boldsymbol{F}^{g})$ and compare it with the small strain volume dilation $e = \text{tr}(\boldsymbol{\epsilon})$.
What does this imply in terms of tissue growth?

```
matlab
J = det(F)
dV = dot(dX3,cross(dX2,dX1))
dv = dot(dx3,cross(dx2,dx1))
J_check = dv / dV;
e = trace(epsilon)
```

```
J = 0.8004
J_check = 0.1345 / 0.1608 = 0.8004
e = -0.1714
```


kinematics of cardiac growth

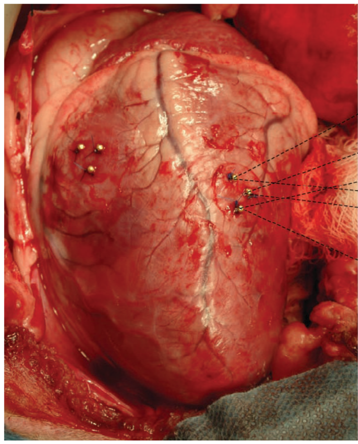


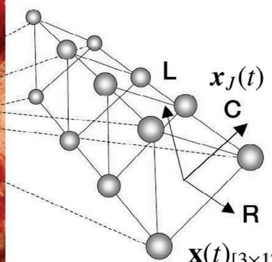
surgically implantation of 4x3 beads across the left ventricular wall

4d coordiantes from in vivo biplane videofluoroscopic marker images

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

kinematics of cardiac growth





deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

valid for all data points

$$\boldsymbol{x}_J(t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X}_J)$$

with coordinates

$$\boldsymbol{x} \;=\; [x_c, x_l, x_r]^{\text{t}}$$

system for 12 markers

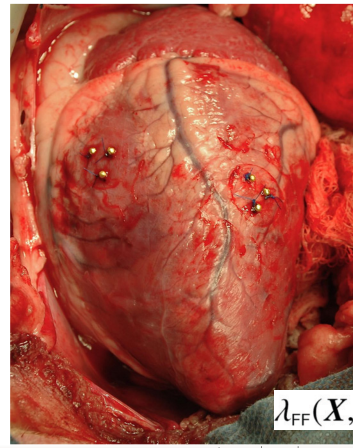
$$\mathbf{x}(t)_{[3\times12]} = \mathbf{c}(t)_{[3\times9]} \cdot \mathbf{N}_{[9\times12]}$$

pseudo inverse to determine coefficients

$$\mathbf{c}(t)_{[3\times9]} = \mathbf{x}(t)_{[3\times12]} \cdot \mathbf{N}^{\text{t}}_{[12\times9]} \cdot [\,\mathbf{N}_{[9\times12]} \cdot \mathbf{N}^{\text{t}}_{[12\times9]}\,]^{-1}$$

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

kinematics of cardiac growth



deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

deformation gradient

$$\boldsymbol{F}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t) \otimes \nabla N_I(\boldsymbol{X})$$

spatial gradient

$$\nabla(\circ) \;=\; [\partial_c(\circ), \partial_l(\circ), \partial_r(\circ)]^{\text{t}}$$

volume changes

$$J(\boldsymbol{X},t) = \det(\boldsymbol{F}(\boldsymbol{X},t))$$

fiber stretch

$$\lambda_{\text{FF}}(\boldsymbol{X},t) = [\,\boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{F}^{\text{t}}(\boldsymbol{X},t) \cdot \boldsymbol{F}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})\,]^{1/2}$$

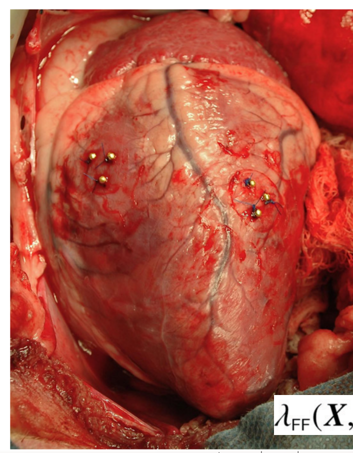
tsamis, cheng, nguyen, langer, miller, kuhl [2012]

kinematics of cardiac growth

	epi 20% depth	p	mid 50% depth	p	endo 80% depth	p
F^g_{CC}	1.00±0.12	0.96	1.03±0.14	0.46	1.02±0.10	0.44
F^g_{CL}	0.04±0.14	0.42	0.01±0.10	0.77	0.01±0.09	0.61
F^g_{CR}	−0.07±0.29	0.46	−0.03±0.16	0.61	0.05±0.14	0.29
F^g_{LC}	−0.02±0.17	0.75	−0.04±0.13	0.33	−0.04±0.11	0.24
F^g_{LL}	1.10±0.15	0.06	1.10±0.13	0.03	1.11±0.11	0.01
F^g_{LR}	0.02±0.16	0.71	0.10±0.20	0.11	0.18±0.34	0.12
F^g_{RC}	−0.01±0.09	0.64	−0.03±0.17	0.54	−0.05±0.19	0.41
F^g_{RL}	0.00±0.05	0.86	−0.00±0.09	0.96	−0.01±0.11	0.67
F^g_{RR}	0.68±0.15	0.00	0.73±0.15	0.00	0.77±0.22	0.01
J^g	0.74±0.19	0.00	0.82±0.19	0.01	0.89±0.21	0.10
λ^g_{FF}	1.03±0.12	0.49	1.04±0.16	0.36	1.08±0.11	0.04

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

kinematics of cardiac growth



deformation

$$\boldsymbol{\varphi}(\boldsymbol{X},t) = \sum_{I=1}^{n_{\text{apx}}} \boldsymbol{c}_I(t)\, N_I(\boldsymbol{X})$$

green lagrange strains

$$\boldsymbol{E}(\boldsymbol{X},t) = \tfrac{1}{2}\,[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} - \boldsymbol{I}\,]$$

fiber strain

$$\text{E}_{\text{FF}}(\boldsymbol{X},t) = \boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{E}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})$$

relation of fiber strain to fiber stretch

$$\text{E}_{\text{FF}} = 1/2\;[\;\lambda_{\text{FF}}^2\; -1\;]$$

fiber stretch

$$\lambda_{\text{FF}}(\boldsymbol{X},t) = [\,\boldsymbol{f}(\boldsymbol{X}) \cdot \boldsymbol{F}^{\text{t}}(\boldsymbol{X},t) \cdot \boldsymbol{F}(\boldsymbol{X},t) \cdot \boldsymbol{f}(\boldsymbol{X})\,]^{1/2}$$

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

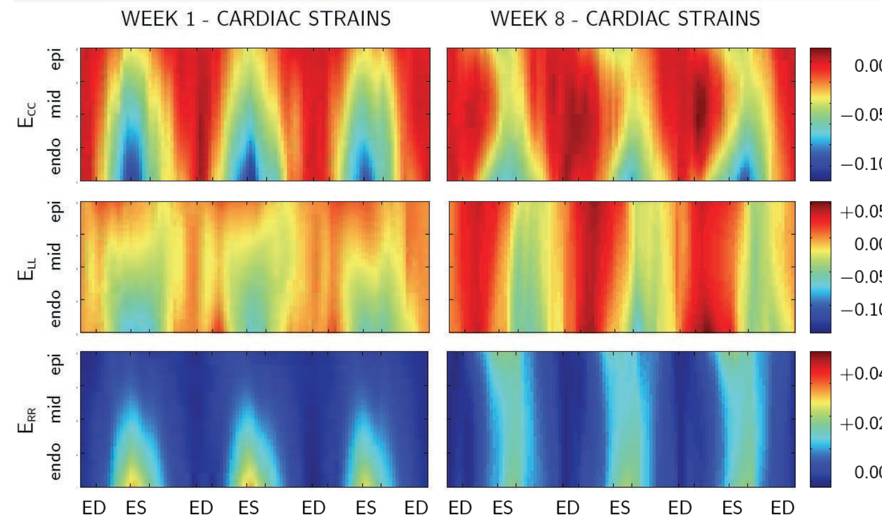
kinematics of cardiac growth

- longitudinal growth by more than 10%
- radial thinning by more than 20%
- fiber lengthening by more than 5%
- volume decrease by more than 15%

	epi 20% depth	p	mid 50% depth	p	endo 80% depth	p
E^g_{CC}	0.03±0.15	0.56	0.06±0.18	0.27	0.05±0.13	0.20
E^g_{LL}	0.12±0.17	0.04	0.12±0.15	0.03	0.13±0.12	0.00
E^g_{RR}	−0.21±0.12	0.00	−0.19±0.09	0.00	−0.10±0.15	0.05
E^g_{CL}	0.01±0.15	0.79	−0.01±0.09	0.63	−0.01±0.06	0.46
E^g_{LR}	0.00±0.08	0.86	0.06±0.11	0.10	0.11±0.19	0.10
E^g_{CR}	−0.04±0.17	0.51	−0.03±0.11	0.39	0.00±0.10	0.88
E^g_{FF}	0.03±0.13	0.42	0.06±0.18	0.31	0.09±0.12	0.03

tsamis, cheng, nguyen, langer, miller, kuhl [2012]

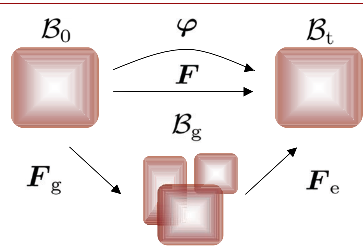
example - growth of the heart



tsamis, cheng, nguyen, langer, miller, kuhl [2012]

example - growth of the heart

kinematics of finite growth

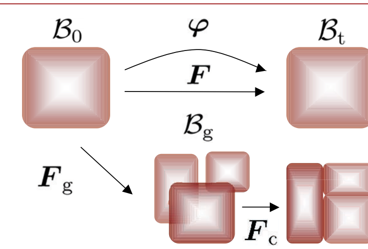


[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[4] loading generates compatible current configuration $\mathcal{B}_t$

concept of residual stress

kinematics of finite growth



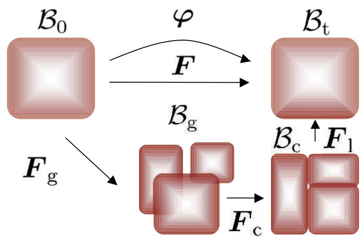
[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

[4] to generate the compatible current configuration $\mathcal{B}_t$

concept of residual stress

kinematics of finite growth



[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

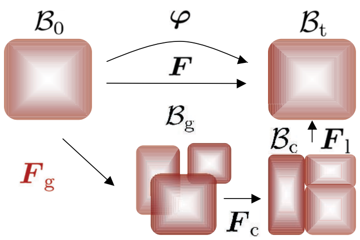
[3b] and then load the compatible configuration $\mathcal{B}_c$ by $\boldsymbol{F}_l$

[4] to generate the compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_l \cdot \boldsymbol{F}_c \cdot \boldsymbol{F}_g$$

growth tensor



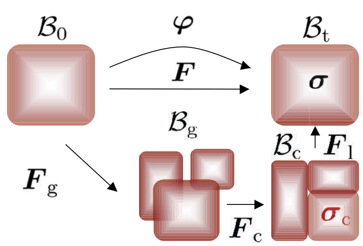
$$\boldsymbol{F}_e = \boldsymbol{F}_l \cdot \boldsymbol{F}_c$$

multiplicative decomposition

lee [1969], simo [1992], rodriguez, hoger & mc culloch [1994], epstein & maugin [2000], humphrey [2002], ambrosi & mollica [2002], himpel, kuhl, menzel & steinmann [2005]

kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_l \cdot \boldsymbol{F}_c \cdot \boldsymbol{F}_g$$



total stress

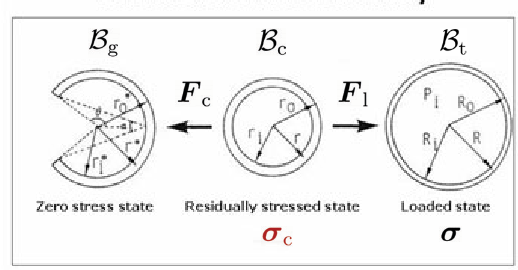
residual stress

residual stress

the additional deformation of squeezing the grown parts back to a compatible configuration gives rise to residual stresses (see thermal stresses)

kinematics of finite growth

Deformation states of an artery



residual stress

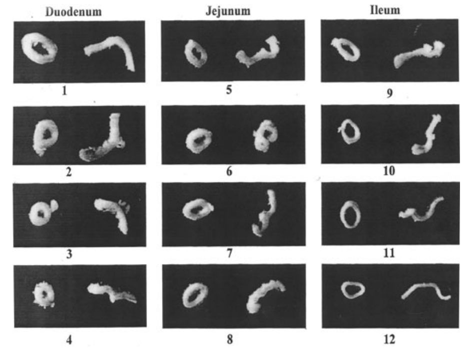
fung [1990], horný, chlup, zitný, mackov [2006]

the classical opening angle experiment

an existence of residual strains in human arteries is well known. it can be observed as an opening up of a circular arterial segment after a radial cut. an opening angle of the arterial segment is used as a measure of the residual strains generally.

fung [1990], horný, chlup, zitný, mackov [2006]

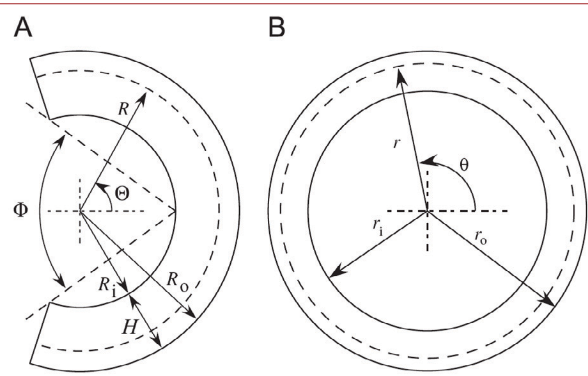
the classical opening angle experiment



photographs showing specimens obtained from different locations in the intestine in the no-load state (left, closed rings) and the zero-stress state (right, open sectors). the rings of jejunum (site 5 to site 8) turned inside out when cut open

zhao, sha, zhuang, gregersen [2002]

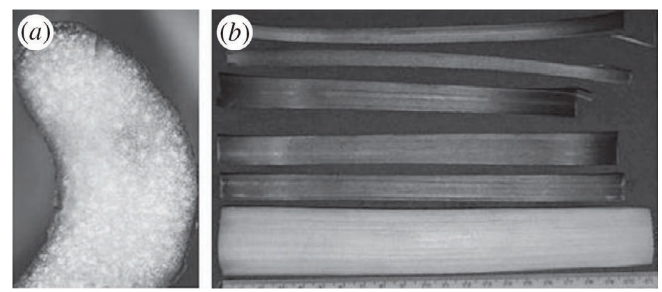
the classical opening angle experiment



global geometrical adaptation - schematic diagram of an arterial cross section in the zero-stress state (A) and in the loaded state (B)

tsamis & stergiopulos [2009]

convince yourself - residual stresses in rhubarb



residual stresses can be easily visualzed in a stalk of rhubarb made up of an outer layer, consisting of epidermal tissue and the collenchyma layers, and an inner layer consisting of parenchyma. when peeled, the **outer strip shortens by -2%** while the **inner layer extends by +6%**. the **inner tissue grows faster** than the outer tissue creating residual stresses resulting from axial tension in the outer wall and axial compression in the inner layer.

atkinson [1900], vandiever & goriely [2009]