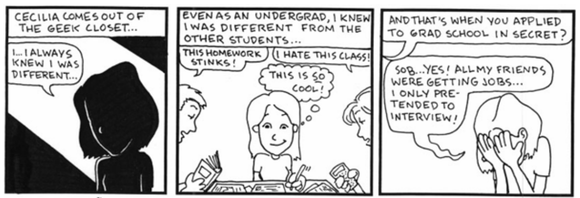
02 - basics and maths - notation and tensors



JORGE CHAM ©THE STANFORD DAILY

day	date		topic
tue	jan	10	motivation - everything grows!
thu	jan	12	basics maths - notation and tensors
tue	jan	17	basic kinematics - large deformation and growth
thu	jan	19	basic kinematics - large deformation and growth
tue	jan	24	guest lecture: class project
thu	jan	26	guest lecture: growing leaflets
tue	jan	31	basic balance equations - closed and open systems
thu	feb	02	basic constitutive equations - growing tumors
tue	feb	07	volume growth - finite elements for growth
thu	feb	09	volume growth - growing arteries
tue	feb	14	volume growth - growing skin
thu	feb	16	volume growth - growing hearts
tue	feb	21	basic constitutive equations - growing bones
thu	feb	23	density growth - finite elements for growth
tue	feb	28	density growth - growing bones
thu	mar	01	everything grows! - midterm summary
tue	mar	06	midterm
thu	mar	08	remodeling - remodeling arteries and tendons
tue	mar	13	class project - discussion, presentation, evaluation
thu	mar	15	class project - discussion, presentation, evaluation
thu	mar	15	written part of final projects due

homework I - first ideas of final project

due 01/19/12, 09:30am, 300-020

Late homework can be dropped off in a box in front of Durand 217. Please mark clearly with date and time @drop off. We will take off 1/10 of points for each 24 hours late, every 12pm after due date. This homework will count 10% towards your final grade.

problem 1 - writing an abstract

The publication *"Perspectives on biological growth and remodeling"* provides a state-of-the-art overview on the mechanics of growth. Read the manuscript carefully. At this point, don't worry if you don't understand all the theoretical details.

The abstract of the publication is pretty poor. Use the sample *"How to construct a Nature summary paragraph"* to rewrite the abstract in your own words. You might not be able to follow the guidelines exactly since this is a review paper without specific results, but try to follow the Nature summary pattern as closely as you can.

nature

How to construct a *Nature* summary paragraph

One or two sentences providing a **basic introduction** to the field, comprehensible to a scientist in any discipline.

Two to three sentences of **more detailed background**, comprehensible to scientists in related disciplines.

One sentence clearly stating the **general problem** being addressed by this particular study.

One sentence summarizing the main result (with the words "**here we show**" or their equivalent).

Two or three sentences explaining what the **main result** reveals in direct comparison to what was thought to be the case previously, or how the main result adds to previous knowledge.

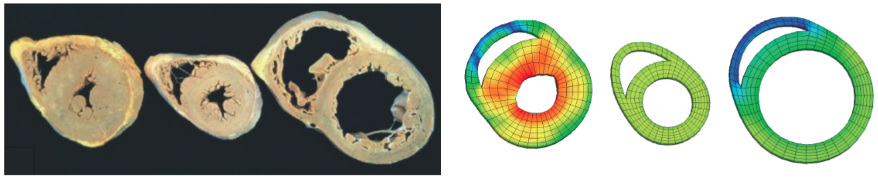
One or two sentences to put the results into a more **general context**.

Two or three sentences to provide a **broader perspective**, readily comprehensible to a scientist in any discipline, may be included in the first paragraph if the editor considers that the accessibility of the paper is significantly enhanced by their inclusion. Under these circumstances, the length of the paragraph can be up to 300 words. (This example is 190 words without the final section, and 250 words with it).

During cell division, mitotic spindles are assembled by microtubule-based motor proteins[1,2]. The bipolar organization of spindles is essential for proper segregation of chromosomes, and requires plus-end-directed homotetrameric motor proteins of the widely conserved kinesin-5 (BimC) family[3]. Hypotheses for bipolar spindle formation include the 'push–pull mitotic muscle' model, in which kinesin-5 and opposing motor proteins act between overlapping microtubules[2,4,5]. However, the precise roles of kinesin-5 during this process are unknown. Here we show that the vertebrate kinesin-5 Eg5 drives the sliding of microtubules depending on their relative orientation. We found in controlled *in vitro* assays that Eg5 has the remarkable capability of simultaneously moving at ~20 nm s^{-1} towards the plus-ends of each of the two microtubules it crosslinks. For anti-parallel microtubules, this results in relative sliding at ~40 nm s^{-1}, comparable to spindle pole separation rates *in vivo*[6]. Furthermore, we found that Eg5 can tether microtubule plus-ends, suggesting an additional microtubule-binding mode for Eg5. Our results demonstrate how members of the kinesin-5 family are likely to function in mitosis, pushing apart interpolar microtubules as well as recruiting microtubules into bundles that are subsequently polarized by relative sliding. We anticipate our assay to be a starting point for more sophisticated *in vitro* models of mitotic spindles. For example, the individual and combined action of multiple mitotic motors could be tested, including minus-end-directed motors opposing Eg5 motility. Furthermore, Eg5 inhibition is a major target of anti-cancer drug development, and a well-defined and quantitative assay for motor function will be relevant for such developments.

problem 2 - identifying things that grow

The publication *"Perspectives on biological growth and remodeling"* provides several examples of biological growth and remodeling. Create a table, which contains three columns: (i) Type of growth; (ii) Biological details; and (iii) Mechanical details. Create entries for as many examples as you can find in this manuscript.



Example. (i) Type of growth: Dilation of the heart; (ii) Biological details: Volume overload induces dilation of the heart through lengthening of the heart muscle cells; (iii) Mechanical details: Growth is anisotropic in the form of lengthening along the muscle fiber direction.

problem 3 - imechanica

Although this publication has been published in 2011, it was written two years before. The web based platform imechanica, http://www.imechanica.org, is a much faster and more interactive medium. Read the January journal club *"Mechanics of Growth"* and the related comments.

Identify more recent examples of growth. Add them to your table. Not all of them are biological, so you may leave out the second column for some examples.

problem 4 - design your own project

Now, it's time to design your own class project. At this point, you don't have to commit to a particular type of growth. This is more meant for you to brainstorm and think of some form of *mechanically driven growth* that you are really excited about. We will carefully read your ideas and give you feedback and literature hints. Describe

- the type of tissue (hard, soft)
- the type of growth (density, surface, volume)
- the level at which you would like to study growth (cellular, tissue, organ)
- the mechanical driving force for growth (strain, stretch, stress, pressure, shear, force)
- the type of adaptation (disease specific, treatment specific, training, ...)
- the way in which you want to study growth (review, analytical, computational, ...)

Write a short summary about what you would like to study, and, if you decide to do this in a group, with whom you would like to work.

tensor calculus

tensor [ˈten.sor] the word tensor was introduced in 1846 by william rowan hamilton. it was used in its current meaning by woldemar voigt in 1899. tensor calculus was developed around 1890 by gregorio ricci-curbastro under the title absolute differential calculus. in the 20th century, the subject came to be known as tensor analysis, and achieved broader acceptance with the introduction of einsteins's theory of general relativity around 1915. tensors are used also in other fields such as continuum mechanics.

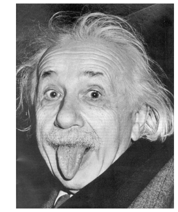
WIKIPEDIA
The Free Encyclopedia

tensor calculus - repetition

- **vector algebra**
  notation, euklidian vector space, scalar product, vector product, scalar triple product
- **tensor algebra**
  notation, scalar products, dyadic product, invariants, trace, determinant, inverse, spectral decomposition, sym-skew decomposition, vol-dev decomposition, orthogonal tensor
- **tensor analysis**
  derivatives, gradient, divergence, laplace operator, integral transformations

vector algebra - notation

- einstein's summation convention

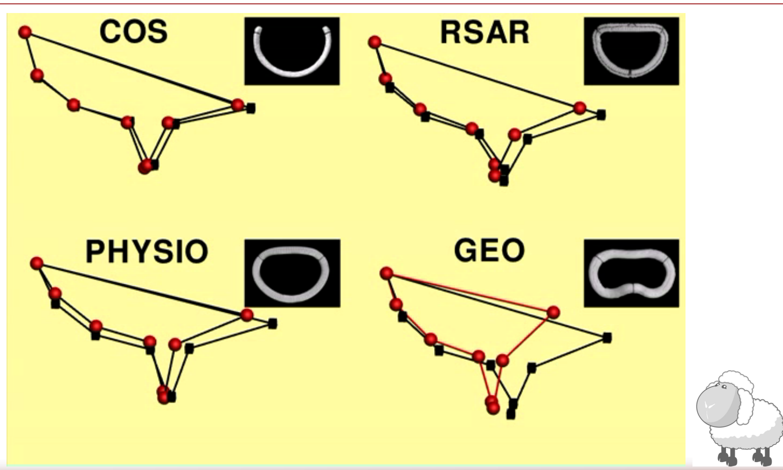


$$u_i = \sum_{j=1}^{3} A_{ij}\, x_j + b_i = A_{ij}\, x_j + b_i$$

- summation over any indices that appear twice in a term

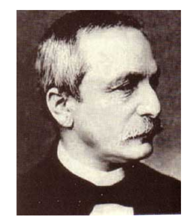
$$\begin{array}{ccccccccc} u_1 & = & A_{11}\, x_1 & + & A_{12}\, x_2 & + & A_{13}\, x_3 & + & b_1 \\ u_2 & = & A_{21}\, x_1 & + & A_{22}\, x_2 & + & A_{23}\, x_3 & + & b_2 \\ u_3 & = & A_{31}\, x_1 & + & A_{32}\, x_2 & + & A_{33}\, x_3 & + & b_3 \end{array}$$

example - position vector / displacement vector



vector algebra - notation

- kronecker symbol



$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases}$$

$$u_i = \delta_{ij}\, u_j$$

- permutation symbol

$$\overset{3}{e}_{ijk} = \begin{cases} 1 & \text{for} \quad \{i,j,k\} \quad \text{... even permutation} \\ -1 & \text{for} \quad \{i,j,k\} \quad \text{... odd permutation} \\ 0 & \qquad\qquad\qquad \text{... else} \end{cases}$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{V}^3$

$$\alpha, \beta \ \in \mathcal{R} \qquad \mathcal{R} \ \ldots \ \text{real numbers}$$
$$\boldsymbol{u}, \boldsymbol{v} \ \in \mathcal{V}^3 \qquad \mathcal{V}^3 \ \ldots \ \text{linear vector space}$$

- $\mathcal{V}^3$ is defined through the following axioms

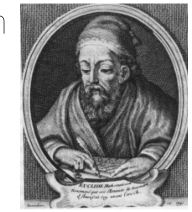
$$\begin{aligned} \alpha\,(\boldsymbol{u}+\boldsymbol{v}) &= \alpha\,\boldsymbol{u}+\alpha\,\boldsymbol{v} \\ (\alpha+\beta)\,\boldsymbol{u} &= \alpha\,\boldsymbol{u}+\beta\,\boldsymbol{u} \\ (\alpha\,\beta)\,\boldsymbol{u} &= \alpha\,(\beta\,\boldsymbol{u}) \end{aligned}$$

- zero element and identity

$$0\,\boldsymbol{u} = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u}$$

- linear independence of $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to $\alpha_i\,\boldsymbol{e}_i = 0$

vector algebra - euklidian vector space



- euklidian vector space $\mathcal{V}^3$equipped with norm

$$n: \ \mathcal{V}^3 \to \mathcal{R} \qquad \ldots \ \text{norm}$$

- norm defined through the following axioms

$$n(\boldsymbol{u}) \geq 0 \qquad n(\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$$
$$n(\alpha\,\boldsymbol{u}) = |\alpha|\,n(\boldsymbol{u})$$
$$n(\boldsymbol{u}+\boldsymbol{v}) \leq n(\boldsymbol{u}) + n(\boldsymbol{v})$$
$$n^2(\boldsymbol{u}+\boldsymbol{v}) + n^2(\boldsymbol{u}-\boldsymbol{v}) = 2\,[n^2(\boldsymbol{u}) + n^2(\boldsymbol{v})]$$

vector algebra - euklidian vector space

- euklidian vector space $\mathcal{E}^3$equipped with euklidian norm

$$n: \ \mathcal{E}^3 \to \mathcal{R} \qquad \ldots \ \text{euklidian norm}$$
$$n(\boldsymbol{u}) = ||\boldsymbol{u}|| = \sqrt{\boldsymbol{u}\cdot\boldsymbol{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2}$$

- representation of 3d vector $\boldsymbol{u} \in \mathcal{E}^3$

$$\boldsymbol{u} = u_i\,\boldsymbol{e}_i = u_1\,\boldsymbol{e}_1 + u_2\,\boldsymbol{e}_2 + u_3\,\boldsymbol{e}_3$$

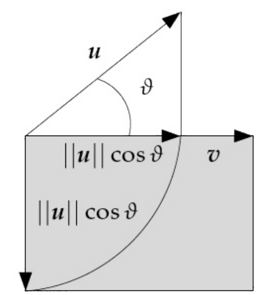
with u_1, u_2, u_3 coordinates (components) of $\boldsymbol{u}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{u} = [\,u_1, u_2, u_3\,]^{\mathrm{t}}$$

vector algebra - scalar product

- euklidian norm enables definition of scalar (inner) product

$$\boldsymbol{u}\cdot\boldsymbol{v} = \alpha \qquad \alpha \in \mathcal{R}$$
$$\boldsymbol{u}\cdot\boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\,\cos\vartheta$$
$$||\boldsymbol{u}\cdot\boldsymbol{v}|| \leq ||\boldsymbol{u}||\,||\boldsymbol{v}||$$

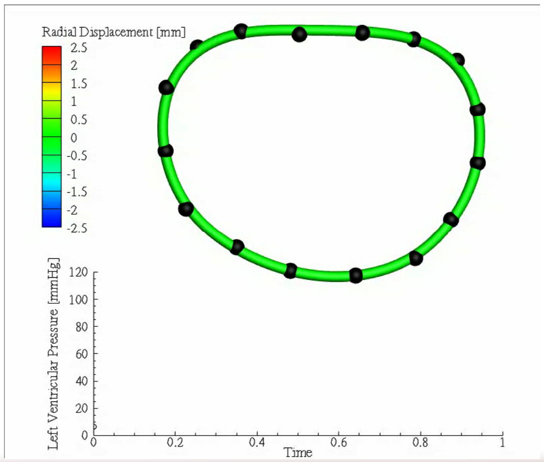


- properties of scalar product

$$\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u}$$
$$(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v})\cdot\boldsymbol{w} = \alpha\,(\boldsymbol{u}\cdot\boldsymbol{w}) + \beta\,(\boldsymbol{v}\cdot\boldsymbol{w})$$
$$\boldsymbol{w}\cdot(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) = \alpha\,(\boldsymbol{w}\cdot\boldsymbol{u}) + \beta\,(\boldsymbol{w}\cdot\boldsymbol{v})$$

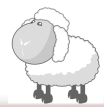
- positive definiteness $\quad \boldsymbol{u}\cdot\boldsymbol{u} \geq 0, \quad \boldsymbol{u}\cdot\boldsymbol{u} = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$
- orthogonality $\quad \boldsymbol{u}\cdot\boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{u} \perp \boldsymbol{v}$

example - radial displacement



$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$

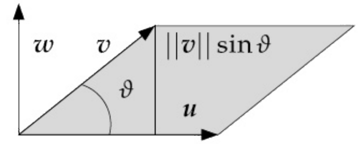
$u_r = \boldsymbol{u} \cdot \boldsymbol{r}$



vector algebra - vector product

- vector product

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \qquad \boldsymbol{w} \in \mathcal{E}^3$$
$$\boldsymbol{u} \times \boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\,\sin\vartheta\,\boldsymbol{n}$$
$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0} \quad \Leftrightarrow \quad \boldsymbol{u} \parallel \boldsymbol{v}$$
$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2\,v_3 & - & u_3\,v_2 \\ u_3\,v_1 & - & u_1\,v_3 \\ u_1\,v_2 & - & u_2\,v_1 \end{bmatrix}$$



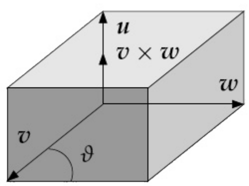
- properties of vector product

$$\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$$
$$(\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) \times \boldsymbol{w} = \alpha\,(\boldsymbol{u} \times \boldsymbol{w}) + \beta\,(\boldsymbol{v} \times \boldsymbol{w})$$
$$\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0$$
$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{u})\,(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2$$

vector algebra - scalar triple product

- scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \alpha \quad \alpha \in \mathcal{R}$$
$$\boldsymbol{u} \times \boldsymbol{v} = ||\boldsymbol{u}||\,||\boldsymbol{v}||\,\sin\vartheta\,\boldsymbol{n} \quad \text{area}$$
$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \quad \text{volume}$$



$$\alpha = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_3 w_1)$$

- properties of scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}] = [\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}]$$
$$= -[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}]$$
$$[\alpha\boldsymbol{u} + \beta\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}] = \alpha\,[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{d}] + \beta\,[\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}]$$

- linear independency $\quad [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \neq 0$

tensor algebra - second order tensors

- second order tensor

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} \qquad \boldsymbol{u} = u_i\,\boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{v} = v_j\,\boldsymbol{e}_j$$
$$\boldsymbol{A} = A_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

with $A_{ij} = u_i\,v_j$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- transpose of second order tensor

$$\boldsymbol{A}^{\mathrm{t}} = (\boldsymbol{u} \otimes \boldsymbol{v})^{\mathrm{t}} = \boldsymbol{v} \otimes \boldsymbol{u}$$
$$\boldsymbol{A}^{\mathrm{t}} = A_{ji}\,\boldsymbol{e}_j \otimes \boldsymbol{e}_i \qquad [A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

tensor algebra - second order tensors

- second order unit tensor in terms of kronecker symbol

$$\boldsymbol{I} = \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

with δ_{ij} coordinates (components) of $\boldsymbol{I}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

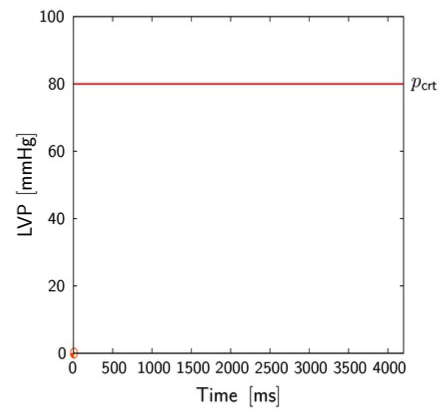
- matrix representation of coordinates

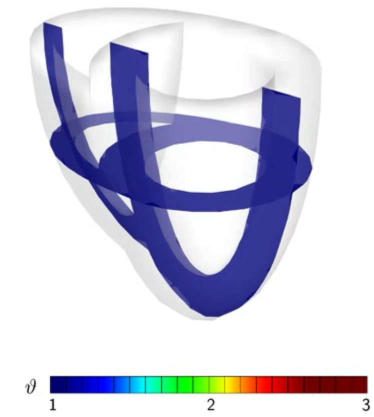
$$[\delta_{ji}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- identity

$$\boldsymbol{I} \cdot \boldsymbol{u} = \boldsymbol{u} \qquad \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot u_j\, \boldsymbol{e}_j = u_i\, \boldsymbol{e}_i$$

example - scaled identity tensor / pressure





tensor algebra - third order tensors

- third order tensor

$$\overset{3}{\boldsymbol{a}} = \boldsymbol{A} \otimes \boldsymbol{v} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{v} = v_k\, \boldsymbol{e}_k$$

$$\overset{3}{\boldsymbol{a}} = a_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

with $a_{ijk} = A_{ij}\, v_k$ coordinates (components) of $\boldsymbol{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- third order permutation tensor in terms of permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{\boldsymbol{e}} = e_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$$

tensor algebra - fourth order tensors

- fourth order tensor

$$\mathbf{A} = \boldsymbol{A} \otimes \boldsymbol{B} \qquad \boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \text{ and } \boldsymbol{B} = B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

$$\mathbf{A} = A_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

with $A_{ijkl} = A_{ij}\, B_{kl}$ coordinates (components) of $\mathbf{A}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

- fourth order unit tensor

$$\mathbf{I} = \delta_{ik}\, \delta_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathbf{I} : \boldsymbol{A} = \boldsymbol{A}$$

- transpose of fourth order unit tensor

$$\mathbf{I}^{\mathrm{t}} = \delta_{il}\, \delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathbf{I}^{\mathrm{t}} : \boldsymbol{A} = \boldsymbol{A}^{\mathrm{t}}$$

tensor algebra - fourth order tensors

- symmetric fourth order unit tensor

$$\mathsf{I}^{\text{sym}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{sym}} : \boldsymbol{A} = \boldsymbol{A}^{\text{sym}}$$

- screw-symmetric fourth order unit tensor

$$\mathsf{I}^{\text{skw}} = \tfrac{1}{2}\,[\delta_{ik}\,\delta_{jl} - \delta_{il}\,\delta_{jk}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{skw}} : \boldsymbol{A} = \boldsymbol{A}^{\text{skw}}$$

- volumetric fourth order unit tensor

$$\mathsf{I}^{\text{vol}} = \tfrac{1}{3}\,\delta_{ij}\,\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \qquad \mathsf{I}^{\text{vol}} : \boldsymbol{A} = \boldsymbol{A}^{\text{vol}}$$

- deviatoric fourth order unit tensor $\quad \mathsf{I}^{\text{dev}} : \boldsymbol{A} = \boldsymbol{A}^{\text{dev}}$

$$\mathsf{I}^{\text{dev}} = [-\tfrac{1}{3}\,\delta_{ij}\,\delta_{kl} + \tfrac{1}{2}\,\delta_{ik}\,\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\,\delta_{jk}]\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{u} &= (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\cdot(u_k\,\boldsymbol{e}_k)\\ &= A_{ij}\,u_k\,\delta_{jk}\,\boldsymbol{e}_i = A_{ij}\,u_j\,\boldsymbol{e}_i = v_i\boldsymbol{e}_i = \boldsymbol{v}\end{aligned}$$

of second order tensor $\boldsymbol{A}$ and vector $\boldsymbol{u}$

- zero and identity $\quad \boldsymbol{0}\cdot\boldsymbol{u} = \boldsymbol{0} \qquad \boldsymbol{I}\cdot\boldsymbol{u} = \boldsymbol{u}$
- positive definiteness $\quad \boldsymbol{a}\cdot\boldsymbol{A}\cdot\boldsymbol{a} > 0$
- properties of scalar product

$$\boldsymbol{A}\cdot(\alpha\boldsymbol{a} + \beta\boldsymbol{b}) = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a}) + \beta\,(\boldsymbol{A}\cdot\boldsymbol{b})$$
$$(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{a} = \boldsymbol{A}\cdot\boldsymbol{a} + \boldsymbol{B}\cdot\boldsymbol{a}$$
$$(\alpha\,\boldsymbol{A})\cdot\boldsymbol{a} = \alpha\,(\boldsymbol{A}\cdot\boldsymbol{a})$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\,\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\\ &= A_{ij}\,B_{kl}\,\delta_{jk}\boldsymbol{e}_i \otimes \boldsymbol{e}_l\\ &= A_{ij}\,B_{jl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_l = C_{il}\boldsymbol{e}_i \otimes \boldsymbol{e}_l = \boldsymbol{C}\end{aligned}$$

of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$

- zero and identity $\quad \boldsymbol{0}\cdot\boldsymbol{A} = \boldsymbol{A} \qquad \boldsymbol{I}\cdot\boldsymbol{A} = \boldsymbol{A}$
- properties of scalar product $\quad (\boldsymbol{A}\cdot\boldsymbol{B})^{\text{t}} = \boldsymbol{B}^{\text{t}}\cdot\boldsymbol{A}^{\text{t}}$

$$\alpha\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (\alpha\,\boldsymbol{A})\cdot\boldsymbol{B} = \boldsymbol{A}\cdot(\alpha\,\boldsymbol{B})$$
$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{C}$$
$$(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot\boldsymbol{C} + \boldsymbol{B}\cdot\boldsymbol{C}$$

tensor algebra - scalar product

- scalar (inner) product

$$\begin{aligned}\boldsymbol{A} : \boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\,\boldsymbol{e}_k \otimes \boldsymbol{e}_l)\\ &= A_{ij}\,B_{kl}\,\delta_{ik}\,\delta_{jl} = A_{ij}\,B_{ij} = \alpha\end{aligned}$$

of two second order tensors $\boldsymbol{A}, \boldsymbol{B}$

- scalar (inner) product

$$\begin{aligned}\mathsf{A} : \boldsymbol{B} &= (A_{ijkl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l) : (B_{mn}\,\boldsymbol{e}_m \otimes \boldsymbol{e}_n)\\ &= A_{ijkl}\,B_{mn}\,\delta_{km}\,\delta_{ln}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\\ &= A_{ijkl}\,B_{kl}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{A}\end{aligned}$$

of fourth order tensors A and second order tensor $\boldsymbol{B}$

- zero and identity $\quad \boldsymbol{0} : \boldsymbol{A} = \boldsymbol{0} \qquad \mathsf{I} : \boldsymbol{A} = \boldsymbol{A}$

tensor algebra - dyadic product

- dyadic (outer) product

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i\, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i\, v_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

of two vectors $\boldsymbol{u}, \boldsymbol{v}$ introduces second order tensor $\boldsymbol{A}$

- properties of dyadic product (tensor notation)

$$\begin{aligned}
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w})\, \boldsymbol{u} \\
&(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) \otimes \boldsymbol{w} = \alpha\, (\boldsymbol{u} \otimes \boldsymbol{w}) + \beta\, (\boldsymbol{v} \otimes \boldsymbol{w}) \\
&\boldsymbol{u} \otimes (\alpha\, \boldsymbol{v} + \beta\, \boldsymbol{w}) = \alpha\, (\boldsymbol{u} \otimes \boldsymbol{v}) + \beta\, (\boldsymbol{u} \otimes \boldsymbol{w}) \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{v} \cdot \boldsymbol{w})\, (\boldsymbol{u} \otimes \boldsymbol{x}) \\
&\boldsymbol{A} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A} \cdot \boldsymbol{u}) \otimes \boldsymbol{v} \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{A} = \boldsymbol{u} \otimes (\boldsymbol{A}^{\mathrm{t}} \cdot \boldsymbol{v})
\end{aligned}$$

tensor algebra - dyadic product

- dyadic (outer) product

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i\, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i\, v_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$$

of two vectors $\boldsymbol{u}, \boldsymbol{v}$ introduces second order tensor $\boldsymbol{A}$

- properties of dyadic product (index notation)

$$\begin{aligned}
&(u_i\, v_j)\, w_j = (v_j\, w_j)\, u_i \\
&(\alpha\, u_i + \beta\, v_i) w_j = \alpha\, (u_i\, w_j) + \beta\, (v_i\, w_j) \\
&u_i (\alpha\, v_j + \beta\, w_j) = \alpha\, (u_i\, v_j) + \beta\, (u_i\, w_j) \\
&(u_i\, v_j)(w_j\, x_k) = (v_j\, w_j)\, (u_i x_k) \\
&A_{ij} (u_j\, v_k) = (A_{ij}\, u_i) v_k \\
&(u_i\, v_j) A_{jk} = u_i (A_{kj}\, v_j)
\end{aligned}$$

tensor algebra - invariants

- (principal) invariants of second order tensor

$$\begin{aligned}
I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\
II_A &= \tfrac{1}{2}\, [\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)] \\
III_A &= \det\,(\boldsymbol{A})
\end{aligned}$$

- derivatives of invariants wrt second order tensor

$$\begin{aligned}
\partial_{\boldsymbol{A}}\ I_A &= \boldsymbol{I} \\
\partial_{\boldsymbol{A}}\ II_A &= I_A\, \boldsymbol{I} - \boldsymbol{A} \\
\partial_{\boldsymbol{A}}\ III_A &= III_A\, \boldsymbol{A}^{-\mathrm{t}}
\end{aligned}$$

tensor algebra - trace

- trace of second order tensor $\quad \mathrm{tr}\,(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v}$

$$\begin{aligned}
\mathrm{tr}(\boldsymbol{A}) &= \mathrm{tr}(A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\
&= A_{ij}\, \mathrm{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j \\
&= A_{ij}\, \delta_{ij} = A_{ii} = A_{11} + A_{22} + A_{33}
\end{aligned}$$

- properties of traces of second order tensors

$$\begin{aligned}
&\mathrm{tr}\,(\boldsymbol{I}) = 3 \\
&\mathrm{tr}\,(\boldsymbol{A}^{\mathrm{t}}) = \mathrm{tr}\,(\boldsymbol{A}) \\
&\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathrm{tr}\,(\boldsymbol{B} \cdot \boldsymbol{A}) \\
&\mathrm{tr}\,(\alpha\, \boldsymbol{A} + \beta\, \boldsymbol{B}) = \alpha\, \mathrm{tr}(\boldsymbol{A}) + \beta\, \mathrm{tr}(\boldsymbol{B}) \\
&\mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{t}}) = \boldsymbol{A} : \boldsymbol{B} \\
&\mathrm{tr}\,(\boldsymbol{A}) = \mathrm{tr}\,(\boldsymbol{A} \cdot \boldsymbol{I}) = \boldsymbol{A} : \boldsymbol{I}
\end{aligned}$$

tensor algebra - determinant

- determinant of second order tensor $\quad III_A = \det(\boldsymbol{A})$

$$\begin{aligned}\det(\boldsymbol{A}) &= \det(A_{ij}) = \tfrac{1}{6}\, e_{ijk}\, e_{abc}\, A_{ia}\, A_{jb}\, A_{kc} \\ &= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\ &- A_{11}A_{23}A_{32} - A_{22}A_{31}A_{13} - A_{33}A_{12}A_{21}\end{aligned}$$

- properties of determinants of second order tensors

$$\begin{aligned}&\det(\boldsymbol{I}) = 1 \\ &\det(\boldsymbol{A}^{\mathrm{t}}) = \det(\boldsymbol{A}) \\ &\det(\alpha\,\boldsymbol{A}) = \alpha^3 \det(\boldsymbol{A}) \\ &\det(\boldsymbol{A}\cdot\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B}) \\ &\det(\boldsymbol{u}\otimes\boldsymbol{v}) = 0\end{aligned}$$

tensor algebra - determinant

- determinant defining vector product

$$\boldsymbol{u}\times\boldsymbol{v} = \det\begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3\end{bmatrix} = \begin{bmatrix} u_2v_3 & - & u_3v_2 \\ u_3v_1 & - & u_1v_3 \\ u_1v_2 & - & u_2v_1\end{bmatrix}$$

- determinant defining scalar triple product

$$[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] = (\boldsymbol{u}\times\boldsymbol{v})\cdot\boldsymbol{w} = \det\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3\end{bmatrix}$$

tensor algebra - inverse

- inverse of second order tensor

$\boldsymbol{A}\cdot\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\cdot\boldsymbol{A} = \boldsymbol{I}$ in particular $\boldsymbol{v} = \boldsymbol{A}\cdot\boldsymbol{u} \quad \boldsymbol{A}^{-1}\cdot\boldsymbol{v} = \boldsymbol{u}$

- adjoint and cofactor

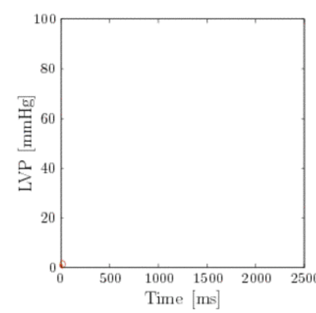
$$\boldsymbol{A}^{\mathrm{adj}} = \det(\boldsymbol{A})\,\boldsymbol{A}^{-1} \qquad \boldsymbol{A}^{\mathrm{cof}} = \det(\boldsymbol{A})\,\boldsymbol{A}^{-\mathrm{t}} = (\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{t}}$$

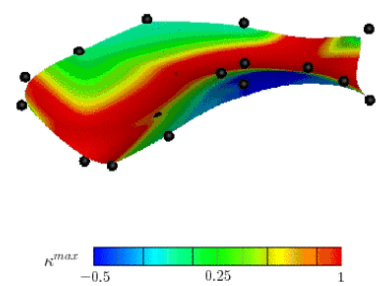
$$\partial_{\boldsymbol{A}}\det(\boldsymbol{A}) = \det(\boldsymbol{A})\,\boldsymbol{A}^{-\mathrm{t}} = III_A\,\boldsymbol{A}^{-\mathrm{t}} = \boldsymbol{A}^{\mathrm{cof}}$$

- properties of inverse $\quad \det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A})$

$$\begin{aligned}(\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \\ (\alpha\,\boldsymbol{A}^{-1})^{-1} &= \alpha^{-1}\,\boldsymbol{A} \\ (\boldsymbol{A}\cdot\boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1}\cdot\boldsymbol{A}^{-1}\end{aligned}$$

eigenvalue problem - maximum principal value





$$\kappa^2 - I_B\kappa + II_B = 0$$

$$\begin{aligned}I_B &:= \mathrm{tr}(\boldsymbol{B}) = B_{\alpha\beta}\,g^{\alpha\beta} = \kappa_1 + \kappa_2 \\ II_B &:= \det(\boldsymbol{B}) = \det(B_{\alpha\beta})/\det(g_{\alpha\beta}) = \kappa_1\kappa_2\end{aligned}$$

tensor algebra - spectral decomposition

- eigenvalue problem of second order tensor

$$\boldsymbol{A}\cdot\boldsymbol{n}_A = \lambda_A\,\boldsymbol{n}_A \qquad [\,\boldsymbol{A}-\lambda_A\,\boldsymbol{I}\,]\cdot\boldsymbol{n}_A=\boldsymbol{0}$$

- solution $\det(\boldsymbol{A}-\lambda_A\,\boldsymbol{I})=0$ in terms of scalar triple product

$$[\boldsymbol{A}\cdot\boldsymbol{u}-\lambda_A\boldsymbol{u},\,\boldsymbol{A}\cdot\boldsymbol{v}-\lambda_A\boldsymbol{v},\,\boldsymbol{A}\cdot\boldsymbol{w}-\lambda_A\boldsymbol{w}]=0$$

- characteristic equation

$$\lambda_A^3 - I_A\,\lambda_A^2 + II_A\,\lambda_A - III_A = 0$$

$$\begin{aligned} I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\ II_A &= \tfrac{1}{2}\,[\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)] \\ III_A &= \det\,(\boldsymbol{A}) \end{aligned}$$

- spectral decomposition

$$\boldsymbol{A} = \textstyle\sum_{i=1}^{3}\lambda_{Ai}\,\boldsymbol{n}_{Ai}\otimes\boldsymbol{n}_{Ai}$$

- cayleigh hamilton theorem

$$\boldsymbol{A}^3 - I_A\,\boldsymbol{A}^2 + II_A\,\boldsymbol{A} - III_A\,\boldsymbol{I} = \boldsymbol{0}$$

tensor algebra - sym/skw decomposition

- symmetric - skew-symmetric decomposition

$$\boldsymbol{A} = \tfrac{1}{2}[\boldsymbol{A}+\boldsymbol{A}^{\mathrm{t}}] + \tfrac{1}{2}[\boldsymbol{A}-\boldsymbol{A}^{\mathrm{t}}] = \boldsymbol{A}^{\mathrm{sym}} + \boldsymbol{A}^{\mathrm{skw}}$$

- symmetric and skew-symmetric tensor

$$\boldsymbol{A}^{\mathrm{sym}} = (\boldsymbol{A}^{\mathrm{sym}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{skw}} = -(\boldsymbol{A}^{\mathrm{skw}})^{\mathrm{t}}$$

- symmetric tensor

$$\boldsymbol{A}^{\mathrm{sym}} = \tfrac{1}{2}[\boldsymbol{A}+\boldsymbol{A}^{\mathrm{t}}] = \mathsf{I}^{\mathrm{sym}} : \boldsymbol{A}$$

- skew-symmetric tensor

$$\boldsymbol{A}^{\mathrm{skw}} = \tfrac{1}{2}[\boldsymbol{A}-\boldsymbol{A}^{\mathrm{t}}] = \mathsf{I}^{\mathrm{skw}} : \boldsymbol{A}$$

tensor algebra - symmetric tensor

- symmetric second order tensor

$$\boldsymbol{A}^{\mathrm{sym}} = \tfrac{1}{2}[\boldsymbol{A}+\boldsymbol{A}^{\mathrm{t}}] \qquad \boldsymbol{A}^{\mathrm{sym}} = (\boldsymbol{A}^{\mathrm{sym}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{sym}} = \boldsymbol{S}$$

- processes three real eigenvalues and corresp.eigenvectors

$$\boldsymbol{S} = \textstyle\sum_{i=1}^{3}\lambda_{Si}\,(\boldsymbol{n}_{Si}\otimes\boldsymbol{n}_{Si}) \qquad \begin{aligned} I_S &= \lambda_{S1}+\lambda_{S2}+\lambda_{S3} \\ II_S &= \lambda_{S2}\,\lambda_{S3}+\lambda_{S3}\,\lambda_{S1}+\lambda_{S1}\,\lambda_{S2} \\ III_S &= \lambda_{S1}\,\lambda_{S2}\,\lambda_{S3} \end{aligned}$$

- square root, inverse, exponent and log

$$\begin{aligned} \sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3}\sqrt{\lambda_{Si}}\,(\boldsymbol{n}_{Si}\otimes\boldsymbol{n}_{Si}) \\ \boldsymbol{S}^{-1} &= \textstyle\sum_{i=1}^{3}\lambda_{Si}^{-1}\,(\boldsymbol{n}_{Si}\otimes\boldsymbol{n}_{Si}) \\ \exp\,(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3}\exp\,(\lambda_{Si})\,(\boldsymbol{n}_{Si}\otimes\boldsymbol{n}_{Si}) \\ \ln\,(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3}\ln\,(\lambda_{Si})\,(\boldsymbol{n}_{Si}\otimes\boldsymbol{n}_{Si}) \end{aligned}$$

tensor algebra - skew-symmetric tensor

- skew-symmetric second order tensor

$$\boldsymbol{A}^{\mathrm{skw}} = \tfrac{1}{2}[\boldsymbol{A}-\boldsymbol{A}^{\mathrm{t}}] \qquad \boldsymbol{A}^{\mathrm{skw}} = -(\boldsymbol{A}^{\mathrm{skw}})^{\mathrm{t}} \qquad \boldsymbol{A}^{\mathrm{skw}} = \boldsymbol{W}$$

- processes three independent entries defining axial vector

$\boldsymbol{w} = -\tfrac{1}{2}\,\overset{3}{\boldsymbol{e}} : \boldsymbol{W} \qquad \boldsymbol{w} = -\,\overset{3}{\boldsymbol{e}}\cdot\boldsymbol{w}$ such that $\boldsymbol{W}\cdot\boldsymbol{v} = \boldsymbol{w}\times\boldsymbol{v}$

- invariants of skew-symmetric tensor

$$\begin{aligned} I_W &= \mathrm{tr}(\boldsymbol{W}) = 0 \\ II_W &= \boldsymbol{w}\cdot\boldsymbol{w} \\ III_W &= \det(\boldsymbol{W}) = 0 \end{aligned}$$

tensor algebra - vol/dev decomposition

- volumetric - deviatoric decomposition

$$\boldsymbol{A} = \boldsymbol{A}^{\text{vol}} + \boldsymbol{A}^{\text{dev}}$$

- volumetric and deviatoric tensor

$$\text{tr}(\boldsymbol{A}^{\text{vol}}) = \text{tr}(\boldsymbol{A}) \qquad \text{tr}(\boldsymbol{A}^{\text{dev}}) = 0$$

- volumetric tensor

$$\boldsymbol{A}^{\text{vol}} = \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\, \boldsymbol{I} = \mathsf{I}^{\text{vol}} : \boldsymbol{A}$$

- deviatoric tensor

$$\boldsymbol{A}^{\text{dev}} = \boldsymbol{A} - \tfrac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\, \boldsymbol{I} = \mathsf{I}^{\text{dev}} : \boldsymbol{A}$$

tensor algebra - orthogonal tensor

- orthogonal second order tensor $\boldsymbol{Q} \in \text{S0}(3)$

$$\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\text{t}} \quad \Leftrightarrow \quad \boldsymbol{Q}^{\text{t}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\text{t}} = \boldsymbol{I}$$

- decomposition of second order tensor

$$\boldsymbol{A} = \boldsymbol{Q} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{Q}$$

such that $\boldsymbol{a} \cdot \boldsymbol{U} \cdot \boldsymbol{a} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{V} \cdot \boldsymbol{a} \geq 0$

- proper orthogonal tensor $\boldsymbol{Q} \in \text{S0}(3)$ has eigenvalue $\lambda_Q = 1$

$$\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q \qquad \text{with} \qquad [Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\varphi & +\sin\varphi \\ 0 & -\sin\varphi & +\cos\varphi \end{bmatrix}$$

interpretation: finite rotation around axis $\boldsymbol{n}_Q$

tensor analysis - frechet derivative

- consider smooth differentiable scalar field Φ with

scalar argument	$\Phi :$	$\mathcal{R} \rightarrow \mathcal{R};$	$\Phi\,(x) = \alpha$
vector argument	$\Phi :$	$\mathcal{R}^3 \rightarrow \mathcal{R};$	$\Phi\,(\boldsymbol{x}) = \alpha$
tensor argument	$\Phi :$	$\mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R};$	$\Phi\,(\boldsymbol{X}) = \alpha$

- frechet derivative (tensor notation)

scalar argument $\quad \mathrm{D}\,\Phi\,(x) = \dfrac{\partial \Phi(x)}{\partial x} = \partial_x \Phi\,(x)$

vector argument $\quad \mathrm{D}\,\Phi\,(\boldsymbol{x}) = \dfrac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_{\boldsymbol{x}} \Phi\,(\boldsymbol{x})$

tensor argument $\quad \mathrm{D}\,\Phi\,(\boldsymbol{X}) = \dfrac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_{\boldsymbol{X}} \Phi\,(\boldsymbol{X})$

tensor analysis - gateaux derivative

- consider smooth differentiable scalar field Φ with

scalar argument	$\Phi :$	$\mathcal{R} \rightarrow \mathcal{R};$	$\Phi\,(x) = \alpha$
vector argument	$\Phi :$	$\mathcal{R}^3 \rightarrow \mathcal{R};$	$\Phi\,(\boldsymbol{x}) = \alpha$
tensor argument	$\Phi :$	$\mathcal{R}^3 \times \mathcal{R}^3 \rightarrow \mathcal{R};$	$\Phi\,(\boldsymbol{X}) = \alpha$

- gateaux derivative,i.e.,frechet wrt direction (tensor notation)

scalar argument $\quad \mathrm{D}\,\Phi\,(x)\;\; u = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi\,(x + \epsilon\, u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R}$

vector argument $\quad \mathrm{D}\,\Phi\,(\boldsymbol{x}) \cdot \boldsymbol{u} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi\,(\boldsymbol{x} + \epsilon\, \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3$

tensor argument $\quad \mathrm{D}\,\Phi\,(\boldsymbol{X}) : \boldsymbol{U} = \dfrac{\mathrm{d}}{\mathrm{d}\epsilon} \Phi(\boldsymbol{X} + \epsilon\, \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3$

tensor analysis - gradient

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f \; (\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f} \; (\boldsymbol{x})$$

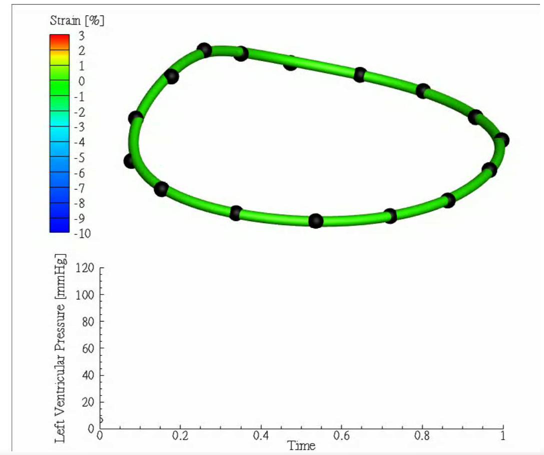
- gradient of scalar- and vector field

$$\nabla f\,(\boldsymbol{x}) = \frac{\partial f\,(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x})\;\boldsymbol{e}_i \qquad \nabla f\,(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix}$$

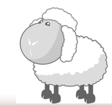
$$\nabla \boldsymbol{f}\,(\boldsymbol{x}) = \frac{\partial f_i\,(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x})\;\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad \nabla \boldsymbol{f}\,(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix}$$

renders vector- and 2nd order tensor field

example - displacement gradient / strain



$$\boldsymbol{\varepsilon} = \nabla \boldsymbol{u}\,(\boldsymbol{x})$$
$$\varepsilon_{\mathrm{tt}} = \boldsymbol{t} \cdot \varepsilon_{\mathrm{t}} \cdot \boldsymbol{t}$$



tensor analysis - divergence

- consider vector- and 2nd order tensor field in domain $\mathcal{B}$

$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f} \; (\boldsymbol{x})$$
$$\boldsymbol{F} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \qquad \boldsymbol{F} : \boldsymbol{x} \to \boldsymbol{F} \; (\boldsymbol{x})$$

- divergence of vector- and 2nd order tensor field

$$\mathrm{div}(\boldsymbol{f}\,(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{f}\,(\boldsymbol{x})) = \nabla \boldsymbol{f}\,(\boldsymbol{x}) : \boldsymbol{I}$$
$$\mathrm{div}(\boldsymbol{f}\,(\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3}$$

$$\mathrm{div}(\boldsymbol{F}\,(\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{F}\,(\boldsymbol{x})) = \nabla \boldsymbol{F}\,(\boldsymbol{x}) : \boldsymbol{I}$$
$$\mathrm{div}(\boldsymbol{F}\,(\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix}$$

renders scalar- and vector field

tensor analysis - laplace operator

- consider scalar- and vector field in domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f \; (\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f} \; (\boldsymbol{x})$$

- laplace operator acting on scalar- and vector field

$$\Delta f\,(\boldsymbol{x}) = \mathrm{div}(\nabla(f\,(\boldsymbol{x}))) \qquad \Delta f\,(\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33}$$

$$\Delta \boldsymbol{f}\,(\boldsymbol{x}) = \mathrm{div}(\nabla(\boldsymbol{f}\,(\boldsymbol{x}))) \qquad \Delta \boldsymbol{f}\,(\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix}$$

renders scalar- and vector field

tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{rll} \alpha: & \mathcal{B} \to \mathcal{R} & \alpha: \boldsymbol{x} \to \alpha \ (\boldsymbol{x}) \\ \boldsymbol{u}: & \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u}: \boldsymbol{x} \to \boldsymbol{u} \ (\boldsymbol{x}) \\ \boldsymbol{v}: & \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{v}: \boldsymbol{x} \to \boldsymbol{v} \ (\boldsymbol{x}) \\ \boldsymbol{A}: & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A}: \boldsymbol{x} \to \boldsymbol{A} \ (\boldsymbol{x}) \end{array}$$

- useful transformation formulae (tensor notation)

$$\begin{aligned} \nabla \ (\alpha \, \boldsymbol{u}) &= \boldsymbol{u} \otimes \nabla \alpha + \alpha \, \nabla \boldsymbol{u} \\ \nabla \ (\boldsymbol{u} \cdot \boldsymbol{v}) &= \boldsymbol{u} \cdot \nabla \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \\ \operatorname{div} \ (\alpha \, \boldsymbol{u}) &= \alpha \operatorname{div}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \alpha \\ \operatorname{div} \ (\alpha \, \boldsymbol{A}) &= \alpha \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} \cdot \nabla \alpha \\ \operatorname{div} \ (\boldsymbol{u} \cdot \boldsymbol{A}) &= \boldsymbol{u} \cdot \operatorname{div}(\boldsymbol{A}) + \boldsymbol{A} : \nabla \boldsymbol{u} \\ \operatorname{div} \ (\boldsymbol{u} \otimes \boldsymbol{v}) &= \boldsymbol{u} \operatorname{div}(\boldsymbol{v}) + \boldsymbol{v} \cdot \nabla \boldsymbol{u}^{\mathrm{t}} \end{aligned}$$

tensor analysis - transformation formulae

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{rll} \alpha: & \mathcal{B} \to \mathcal{R} & \alpha: x_k \to \alpha \ (x_k) \\ u_i: & \mathcal{B} \to \mathcal{R}^3 & u_i: x_k \to u_i \ (x_k) \\ v_i: & \mathcal{B} \to \mathcal{R}^3 & v_i: x_k \to v_i \ (x_k) \\ A_{ij}: & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & A_{ij}: x_k \to A_{ij} \ (x_k) \end{array}$$

- useful transformation formulae (index notation)

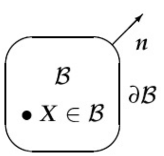
$$\begin{aligned} (\alpha \, u_i)_{,j} &= u_i \, \alpha_{,j} + \alpha \, u_{i,j} \\ (u_i \, v_i)_{,j} &= u_i \, v_{i,j} + v_i \, u_{i,j} \\ (\alpha \, u_i)_{,i} &= \alpha \, u_{i,i} + u_i \, \alpha_{,i} \\ (\alpha \, A_{ij})_{,j} &= \alpha \, A_{ij,j} + A_{ij} \, \alpha_{,j} \\ (u_i \, A_{ij})_{,j} &= u_i \, A_{ij,j} + A_{ij} \, u_{i,j} \\ (u_i \, v_j)_{,j} &= u_i \, v_{j,j} + v_j \, u_{i,j} \end{aligned}$$

tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{rll} \alpha: & \mathcal{B} \to \mathcal{R} & \alpha: \boldsymbol{x} \to \alpha \ (\boldsymbol{x}) \\ \boldsymbol{u}: & \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u}: \boldsymbol{x} \to \boldsymbol{u} \ (\boldsymbol{x}) \\ \boldsymbol{A}: & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A}: \boldsymbol{x} \to \boldsymbol{A} \ (\boldsymbol{x}) \end{array}$$

- integral theorems (tensor notation)



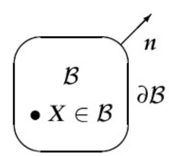
$$\begin{array}{lll} \int_{\partial\mathcal{B}} \alpha \, \boldsymbol{n} \, \mathrm{d}A = \int_{\mathcal{B}} \nabla \alpha \, \mathrm{d}V & & \text{green} \\ \int_{\partial\mathcal{B}} \boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_{\mathcal{B}} \operatorname{div}(\boldsymbol{u}) \, \mathrm{d}V & & \text{gauss} \\ \int_{\partial\mathcal{B}} \boldsymbol{A} \cdot \boldsymbol{n} \, \mathrm{d}A = \int_{\mathcal{B}} \operatorname{div}(\boldsymbol{A}) \, \mathrm{d}V & & \text{gauss} \end{array}$$

tensor analysis - integral theorems

- consider scalar,vector and 2nd order tensor field on $\mathcal{B} \in \mathcal{R}^3$

$$\begin{array}{rll} \alpha: & \mathcal{B} \to \mathcal{R} & \alpha: x_k \to \alpha \ (x_k) \\ u_i: & \mathcal{B} \to \mathcal{R}^3 & u_i: x_k \to u_i \ (x_k) \\ A_{ij}: & \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & A_{ij}: x_k \to A_{ij} \ (x_k) \end{array}$$

- integral theorems (tensor notation)



$$\begin{array}{lll} \int_{\partial\mathcal{B}} \alpha \, n_i \, \mathrm{d}A = \int_{\mathcal{B}} \alpha_{,i} \, \mathrm{d}V & & \text{green} \\ \int_{\partial\mathcal{B}} u_i \, n_i \, \mathrm{d}A = \int_{\mathcal{B}} u_{i,i} \, \mathrm{d}V & & \text{gauss} \\ \int_{\partial\mathcal{B}} A_{ij} \, n_j \, \mathrm{d}A = \int_{\mathcal{B}} A_{ij,j} \, \mathrm{d}V & & \text{gauss} \end{array}$$

voigt / matrix vector notation

- strain tensors as vectors in voigt notation

$$E_{ij} = \begin{bmatrix} E_{11} & E_{12} & E_{31} \\ E_{12} & E_{22} & E_{23} \\ E_{31} & E_{23} & E_{33} \end{bmatrix}$$

$$E^{\text{voigt}} = [E_{11}, E_{22}, E_{33}, 2\,E_{12}, 2\,E_{23}, 2\,E_{31}]^{\text{t}}$$

- stress tensors as vectors in voigt notation

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{31} \\ S_{12} & S_{22} & S_{23} \\ S_{31} & S_{23} & S_{33} \end{bmatrix}$$

$$S^{\text{voigt}} = [S_{11}, S_{22}, S_{33}, S_{12}, S_{23}, S_{31}]^{\text{t}}$$

- why are strain & stress different? check energy expression!

$$\psi = \tfrac{1}{2}\,\boldsymbol{E} : \boldsymbol{S} \qquad \psi = \tfrac{1}{2}\,\boldsymbol{E}^{\text{voigt}} : \boldsymbol{S}^{\text{voigt}}$$

voigt / matrix vector notation

- fourth order material operators as matrix in voigt notation

$$C^{\text{voigt}} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1131} \\ C_{2211} & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2231} \\ C_{3311} & C_{3322} & C_{3333} & C_{3312} & C_{3323} & C_{3331} \\ C_{1211} & C_{1222} & C_{1233} & C_{1212} & C_{1223} & C_{1231} \\ C_{2311} & C_{2322} & C_{2333} & C_{2312} & C_{2323} & C_{2331} \\ C_{3111} & C_{3122} & C_{3133} & C_{3112} & C_{3123} & C_{3331} \end{bmatrix}$$

- why are strain & stress different? check these expressions!

$$\boldsymbol{S} = \mathsf{C} : \boldsymbol{E} \qquad \boldsymbol{S}^{\text{voigt}} = \boldsymbol{C}^{\text{voigt}} \cdot \boldsymbol{E}^{\text{voigt}}$$