mechanics of growth

me 337

ellen kuhl
mechanical engineering
stanford university

… who i am …

since 01/07 - assistant professor me

… who i am …

… what i do …

kinematic equations for finite growth

$$\boldsymbol{F} = \boldsymbol{F}_{\rm e} \cdot \boldsymbol{F}_{\rm g}$$

balance equations for open systems

$$\mathrm{D}_t \rho_0 = \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0$$
$$\rho_0 \, \mathrm{D}_t \boldsymbol{v} = \mathrm{Div}(\boldsymbol{P}) + \boldsymbol{b}_0$$

constitutive equations for living tissues

$$\boldsymbol{P} = \boldsymbol{P}(\rho_0, \boldsymbol{F}, \boldsymbol{F}_{\rm g})$$

fe analyses for biological structures

continuum- & computational biomechanics

… what i do …

… why i do what i do …

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highly deformable

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living

nonlinear

inelastic

kinematic equations for finite growth

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- constitutive equations

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- fe analyses for biological structures

highly deformable

living

anisotropic

inelastic

nonlinear

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- fe analyses for biological structures

highly deformable

living

anisotropic

inelastic

nonlinear

inhomo-geneous

… because biological structures are …

content

me - goals

In contrast to traditional engineering structures living structures show the fascinating ability to grow and adapt their form, shape and microstructure to a given mechanical environment. This course addresses the phenomenon of growth on a theoretical and computational level and applies the resulting theories to classical biomechanical problems like bone remodeling, hip replacement, wound healing, atherosclerosis or in stent restenosis. This course will illustrate how classical engineering concepts like continuum mechanics, thermodynamics or finite element modeling have to be rephrased in the context of growth. Having attended this course, you will be able to develop your own problemspecific finite element based numerical solution techniques and interpret the results of biomechanical simulations with the ultimate goal of improving your understanding of the complex interplay between form and function.

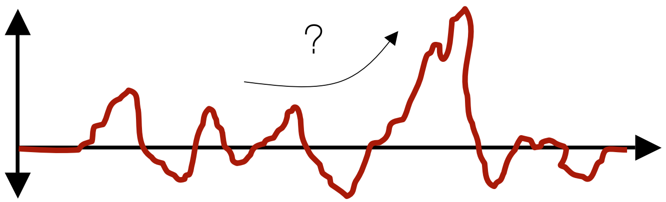
me 337 - syllabus

day	date		topic	Homework
tue	apr	03	introduction - different forms of growth	
thu	apr	05	introduction – history of growth theories	#1 wiki growth
tue	apr	10	kinematic equations – finite growth	
thu	apr	12	balance equations – classical	
tue	apr	17	balance equations – growth	galileo problem
thu	apr	19	constitutive equations – density growth	
tue	apr	24	constitutive equations – volume growth	
thu	apr	26	finite element method – np density theory	
tue	mai	01	finite element method – np density matlab	
thu	mai	03	examples – bone remodeling	#2 example bone
tue	mai	08	finite element method – ip density theory	
thu	mai	10	finite element method – ip density matlab	#3 matlab const
tue	mai	15	finite element method – np vs ip comparison	take-home assign
thu	mai	17	example – hip replacement, wound healing	
tue	mai	22	kinematic equations – volume growth	
thu	mai	24	balance equations – volume growth	galileo problem
tue	mai	29	finite element method - ip volume theory	#4 wiki growth
thu	mai	31	finite element method – ip volume matlab	
tue	jun	05	example – atherosclerosis, in stent restenosis	
thu	jun	07	wiki session – vote on articles	

me 337 - suggested reading

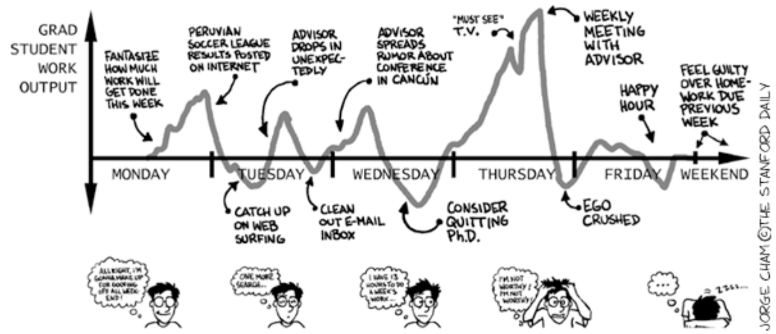
1. LA Taber: "Biomechanics of growth, remodeling and morphogenesis", *Appl. Mech. Rev.*, Vol 48, pp. 487-545, 1995
2. CR Jacobs, ME Levenston, GS Beaupré, JC Simo, DR Carter: "Numerical instabilities in bone remodeling simulations: The advantages of a node-based finite element approach", *Journal of Biomechanics*, Vol 28, pp. 449-459, 1995.
3. E Kuhl, A Menzel, P Steinmann: "Computational modeling of growth: A critical review, a classification of concepts and two new consistent approaches", *Computational Mechanics*, Vol 32, pp. 71-88, 2003.
4. EK Rodriguez, A Hoger, AD Mc~Culloch: "Stress--dependent finite growth in soft elastic tissues", *Journal of Biomechanics*, Vol 27, pp. 455-467,1994.
5. E Kuhl, R Maas, G Himpel, A Menzel: "Computational modeling of arterial wall growth: Attempts towards patient-specific simulations based on computer tomography", *Biomechanics and Modeling in Mechanobiology*, available online, DOI 10.1007/s10237-006-0062-x, 2006.

what's growing?



classical engineering materials are not!

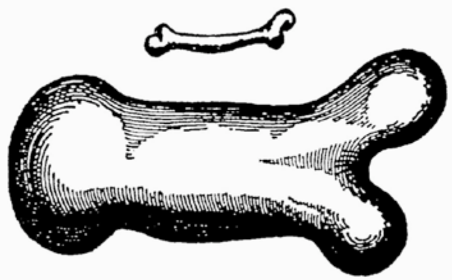
what's growing?



grad student work output ;-)

J. Cham "Piled higher and deeper", [1999]

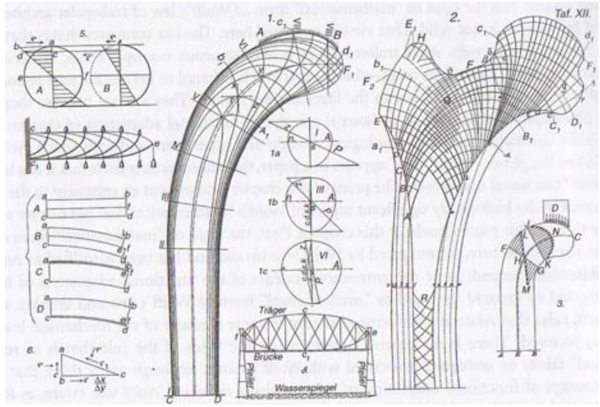
history - 17th century



„…dal che e manifesto, che chi volesse mantener in un vastissimo gigante le proporzioni, che hanno le membra in un huomo ordinario, bisognerebbe o trouar materia molto piu dura, e resistente per formarne l'ossa o vero ammettere, che la robustezza sua fusse a proporzione assai piu fiacca, che negli huomini de statura mediocre; altrimente crescendogli a smisurata altezza si vedrebbono dal proprio peso opprimere, e cadere…"

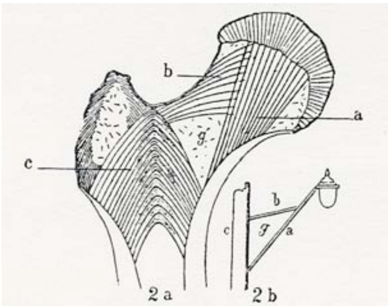
Galileo,"Discorsi e dimostrazioni matematiche", [1638]

history - 19th century



Culmann & von Meyer „Graphic statics" [1867]

history - 19th century



„…es ist demnach unter dem gesetze der transformation der knochen dasjenige gesetz zu verstehen, nach welchem im gefolge primaerer abaenderungen der form und inanspruchnahme bestimmte umwandlungen der inneren architectur und umwandlungen der aeusseren form sich vollziehen…"

Wolff „Gesetz der Transformation der Knochen" [1892]

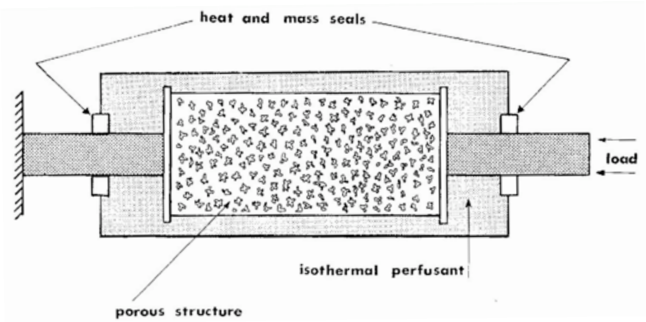
history - 19th century

carson pirie scott store
Sullivan[1904]

„…whether it be the sweeping eagle in his flight or the open apple-blossom, the toiling work-horse, the blithe swan, the branching oak, the winding stream at its base, the drifting clouds, over all the coursing sun, form ever follows function, and this is the law…"

Sullivan „Form follows function" [1896]

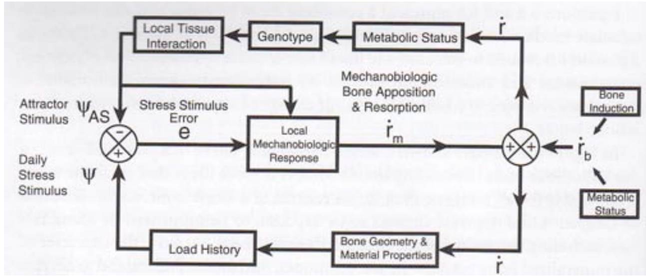
history - 20th century



„…the system consisting of only the porous structure without its entrained perfusant is open with respect to momentum transfer as well as mass, energy, and entropy transfer. we shall write balance and constitutive equations for only the bone…"

Cowin & Hegedus „Theory of adaptive elasticity" [1976]

history - 20th century



„…the relationship between physical forces and the morphology of living things has piqued the curiosity of every artist, scientist, or philosopher who has contemplated a tree or drawn the human figure. its importance was a concern of galileo and later thompson whose writings remind us that physical causation plays an inescapable role in the development of biological form…"

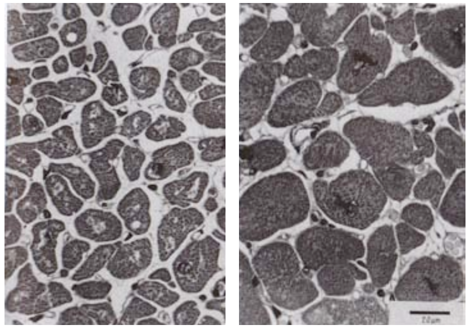
Beaupré, Carter & Orr „Theory of bone modeling & remodeling" [1990]

history - 20th century

„hypertrophy of the heart: comparison of cross sections of a normal heart (bottom), a heart chronically overloaded by an unusually large blood volume (left) and a heart chronically overloaded by an unusually large diastoic and systolic left ventricular pressure (right)"

Fung „Biomechanics - Motion, flow, stress, and growth" [1990]

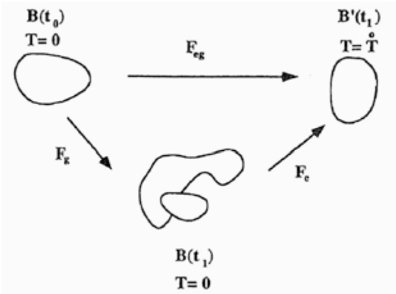
history - 20th century



„hypertrophy of the heart: histology of a normal heart (left) and pressure overloaded heart (right) photographed at the same magnification - muscles in the hypertropic heart (right) are much bigger in diameter than those of the normal heart (left)."

Fung „Biomechanics - Motion, flow, stress, and growth" [1990]

history - 21th century



Rodriguez, Holger & McCulloch [1994]

„...the process of growth can be seen as an evolution of material point neighbourhoods in a fixed reference configuration. the growth process will cause the development of material inhomogeneities responsible for residual stresses in the body..."

Epstein & Maugin „Theory of volumetric growth" [2000]

growth, remodeling and morphogenesis

growth [groʊθ] which is defined as added mass, can occur through cell division (hyperplasia), cell enlargement (hypertrophy), secretion of extracellular matrix, or accretion @ external or internal surfaces. negative growth (atrophy) can occur through cell death, cell shrinkage, or resorption. in most cases, hyperplasia and hypertrophy are mutually exclusive processes. depending on the age of the organism and the type of tissue, one of these two growth processes dominates.

Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

growth, remodeling and morphogenesis

remodeling [riˈmad. l.ɪŋ] involves changes in material properties. These changes, which often are adaptive, may be brought about by alterations in modulus, internal structure, strength, or density. for example, bones, and heart muscle may change their internal structures through reorientation of trabeculae and muscle fibers, respectively.

Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

growth, remodeling and morphogenesis

morphogenesis[mɔːr.foˈdʒen.ə.sɪs]is the generation of animal form. usually, the term refers to embryonic development, but wound healing and organ regeneration are also morphogenetic events. morphogenesis contains a complex series of stages, each of which depends on the previous stage. during these stages, genetic and environmental factors guide the spatial-temporal motions and differentiation (specification) of cells. a flaw in any one stage may lead to structural defects.

Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

mechanics of growth

write a wikipedia article about the **mechanics of growth** [məˈkæn.ɪks ɑːv groʊθ] about one page long. you may use taber's 1995 paper but any other sources are welcome (please cite). you can work in groups of two. at the end of this course, you will revise your article to see how your knowledge about growth has increased. finally, the class will decide which one to post on **http://www.wikipedia.org/**

Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

continuum mechancis

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is a branch of physics (specifically mechanics) that deals with continuous matter. the fact that matter is made of atoms and that it commonly has some sort of heterogeneous microstructure is ignored in the simplifying approximation that physical quantities, such as energy and momentum, can be handled in the infinitesimal limit. differential equations can thus be employed in solving problems in continuum mechanics.

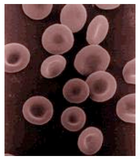
continuum mechanics

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular structure of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions. this hypothetical continuous material we call a continuum.

Malvern „Introduction to the mechanics of a continuous medium" [1969]

continuum mechanics

continuum hypothesis [kənˈtɪn.ju.əm haɪˈpɑːθ.ə.sɪs] we assume that the characteristic length scale of the microstructure is much smaller than the characteristic length scale of the overall problem, such that the properties at each point can be understood as averages over a characteristic length scale



$$l^{\rm micro} \ll l^{\rm averg} \ll l^{\rm conti}$$

example: biomechanics

$$l^{\rm micro} = l^{\rm cells} \approx 10 \mu{\rm m}$$
$$l^{\rm conti} = l^{\rm tissue} \approx 10 {\rm cm}$$

the continuum hypothesis can be applied when analyzing tissues

the potato equations

- kinematic equations - what's strain? $\epsilon = \frac{\Delta l}{l}$
  general equations that characterize the deformation of a physical body without studying its physical cause
- balance equations - what's stress? $\sigma = \frac{F}{A}$
  general equations that characterize the cause of motion of any body
- constitutive equations - how are they related? $\sigma = E\,\epsilon$
  material specific equations that complement the set of governing equations

the potato equations

- kinematic equations - why not $\epsilon = \frac{\Delta l}{l}$?
  inhomogeneous deformation » non-constant
  finite deformation » non-linear $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  inelastic deformation » growth tensor $\boldsymbol{F} = \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{g}}$
- balance equations - why not $\sigma = \frac{F}{A}$? $\mathrm{Div}(\boldsymbol{P}) + \rho\, \boldsymbol{b}_0 = \boldsymbol{0}$
  equilibrium in deformed configuration » multiple stress measures
- constitutive equations - why not $\sigma = E\,\epsilon$?
  finite deformation » non-linear $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  inelastic deformation » internal variables $\boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_{\mathrm{g}})$

kinematic equations

kinematic equations [kɪnəˈmætɪk ɪˈkweɪ.ʒəns] describe the motion of objects without the consideration of the masses or forces that bring about the motion. the basis of kinematics is the choice of coordinates. the 1st and 2nd time derivatives of the position coordinates give the velocities and accelerations. the difference in placement between the beginning and the final state of two points in a body expresses the numerical value of strain. strain expresses itself as a change in size and/or shape.

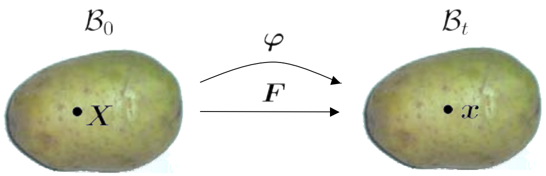
WIKIPEDIA The Free Encyclopedia

kinematic equations

kinematics [kɪnəˈmætɪks] is the study of motion per se, regardless of the forces causing it. the primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.

Chadwick „Continuum mechanics" [1976]

potato - kinematics



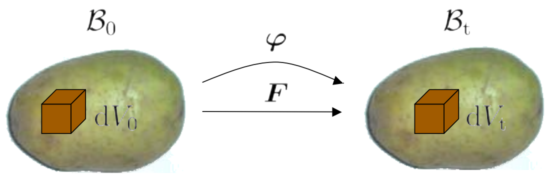
- nonlinear deformation map φ

  $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$ with $\varphi : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$

- spatial derivative of φ - deformation gradient

  $\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X}$ with $\boldsymbol{F} : T\mathcal{B}_0 \to T\mathcal{B}_t$ $\quad \boldsymbol{F} = \left.\frac{\partial \varphi}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$

potato - kinematics of finite growth



- transformation of volume elements - determinant of $\boldsymbol{F}$

  $\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3]$ $\quad \mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3]$

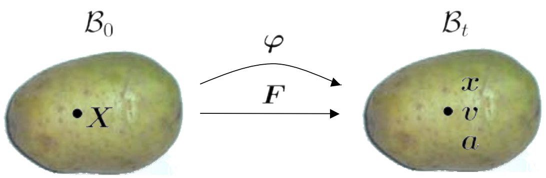
  $= \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3])$

  $= \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$ $\quad = \det(\boldsymbol{F}) \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$

- changes in volume - determinant of deformation tensor J

  $\mathrm{d}V_t = J\, \mathrm{d}V_0$ $\quad J = \det(\boldsymbol{F})$

potato - kinematics



- temporal derivative of φ - velocity (material time derivative)

  $\boldsymbol{v} = \mathrm{D}_t \varphi = \left.\frac{\partial \varphi}{\partial t}\right|_{X \text{ fixed}}$ with $\boldsymbol{v} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$

- temporal derivative of $\boldsymbol{v}$ - acceleration

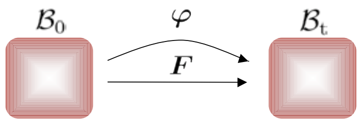
  $\boldsymbol{a} = \mathrm{D}_t \boldsymbol{v} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{X \text{ fixed}} = \left.\frac{\partial^2 \varphi}{\partial t^2}\right|_{X \text{ fixed}}$ with $\boldsymbol{a} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$

volume growth

volume growth [ˈval.ju:m groʊθ] is conceptually comparable to thermal expansion. in linear elastic problems, growth stresses (such as thermal stresses) can be superposed on the mechanical stress field. in the nonlinear problems considered here, another approach must be used. the fundamental idea is to refer the strain measures in the constitutive equations of each material element to its current zero-stress configuration, which changes as the element grows.

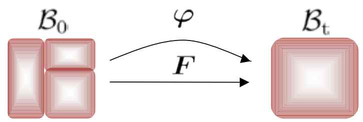
Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded, stressfree

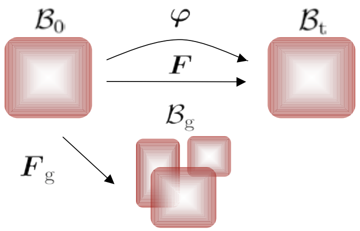
kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded, stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

kinematics of finite growth

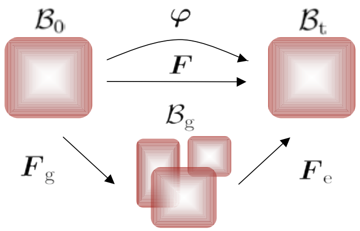


[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded, stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the element may be incompatible $\mathcal{B}_g$

kinematics of finite growth



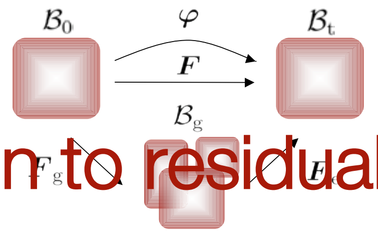
[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded, stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the element may be incompatible $\mathcal{B}_g$

[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth



Relation to residual stress

[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded, stressfree

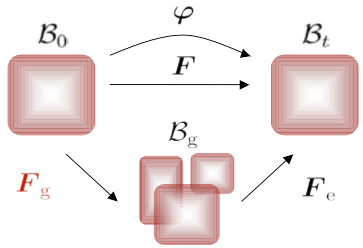
[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

[3] after growing the element may be incompatible $\mathcal{B}_g$

[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

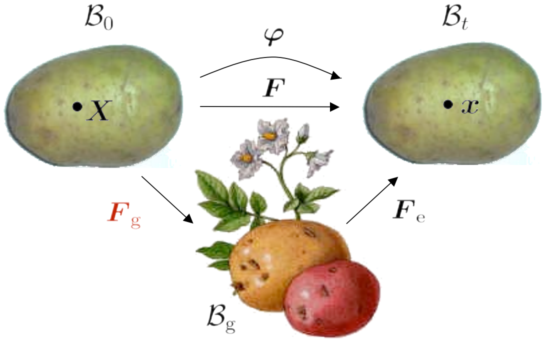
$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$



multiplicative decomposition

Lee [1969], Simo [1992], Rodriguez, Hoger & Mc Culloch [1994], Epstein & Maugin [200
Humphrey [2002], Ambrosi & Mollica [2002], Himpel, Kuhl, Menzel & Steinmann [2005]

potato - kinematics of finite growth

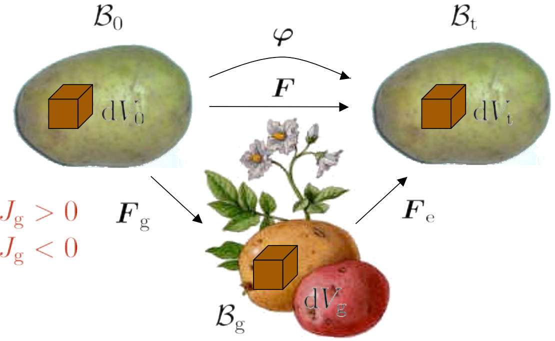


- incompatible growth configuration $\mathcal{B}_g$ & growth tensor $\boldsymbol{F}_g$

$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$

Rodriguez, Hoger & McCulloch [1994]

potato - kinematics of finite growth



- changes in volume - determinant of growth tensor J_g

$$\mathrm{d}V_g = J_g \, \mathrm{d}V_0 \qquad J_g = \det(\boldsymbol{F}_g)$$

balance equations

balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, momentum, angular momentum and energy, supplemented with an entropy inequality constitute the set of conservation laws.
the law of conservation of mass/matter states that the mass of a closed system of substances will remain constant, regardless of the processes acting inside the system.
the principle of conservation of momentum states that the total momentum of a closed system of objects is constant.

Wikipedia The Free Encyclopedia

balance equations

balance equations [ˈbæl.əns ɪˈkweɪ.ʒəns] of mass, linear momentum, angular momentum and energy apply to all material bodies. each one gives rise to a field equation, holing on the configurations of a body in a sufficiently smooth motion and a jump condition on surfaces of discontinuity. like position, time and body, the concepts of mass, force, heating and internal energy which enter into the formulation of the balance equations are regarded as having primitive status in continuum mechanics.

Chadwick „Continuum mechanics" [1976]