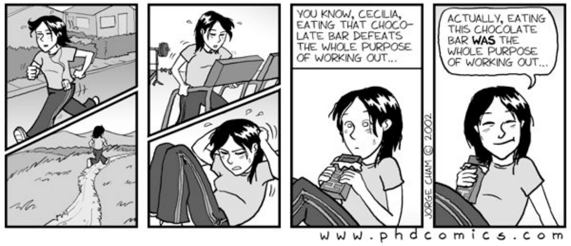
07 - constitutive equations - density growth



constitutive equations

constitutive equations [kən'stɪ.tu.tɪv ɪ'kweɪ.ʒəns] in structural analysis, constitutive relations connect applied stresses or forces to strains or deformations. the constitutive relations for linear materials are linear. more generally, in physics, a constitutive equation is a relation between two physical quantities (often tensors) that is specific to a material, and does not follow directly from physical law. some constitutive equations are simply phenomenological; others are derived from first principles.

Wikipedia The Free Encyclopedia

constitutive equations

constitutive equations [kən'stɪ.tu.tɪv ɪ'kweɪ.ʒəns] or equations of state bring in the characterization of particular materials within continuum mechanics. mathematically, the purpose of these relations is to supply connections between kinematic, mechanical and thermal fields. physically, constitutive equations represent the various forms of idealized material response which serve as models of the behavior of actual substances.

Chadwick „Continuum mechanics" [1976]

tensor analysis - basic derivatives

$$\{\bullet \otimes \circ\}_{ijkl} = \{\bullet\}_{ij}\,\{\circ\}_{kl}$$
$$\{\bullet \overline{\otimes} \circ\}_{ijkl} = \{\bullet\}_{ik}\,\{\circ\}_{jl}$$
$$\{\bullet \underline{\otimes} \circ\}_{ijkl} = \{\bullet\}_{il}\,\{\circ\}_{jk}$$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{F}} = \boldsymbol{I} \overline{\otimes} \boldsymbol{I} \qquad \frac{\partial F_{ij}}{\partial F_{kl}} = \delta_{ik}\,\delta_{jl}$$
$$\frac{\partial \boldsymbol{F}^{-1}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{-1} \overline{\otimes} \boldsymbol{F}^{-t} \qquad \frac{\partial F^{-1}_{ij}}{\partial F_{kl}} = -F^{-1}_{ik}\,F^{-1}_{lj}$$
$$\frac{\partial \boldsymbol{F}^{t}}{\partial \boldsymbol{F}} = \boldsymbol{I} \underline{\otimes} \boldsymbol{I} \qquad \frac{\partial F_{ji}}{\partial F_{kl}} = \delta_{il}\,\delta_{jk}$$
$$\frac{\partial \boldsymbol{F}^{-t}}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{-t} \underline{\otimes} \boldsymbol{F}^{-1} \qquad \frac{\partial F^{-1}_{ji}}{\partial F_{kl}} = -F^{-1}_{li}\,F^{-1}_{jk}$$

tensor analysis - basic derivatives

$$\frac{\partial \det(\boldsymbol{F})}{\partial \boldsymbol{F}} = \det(\boldsymbol{F})\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = \boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \det(\boldsymbol{F}^{\text{-1}})}{\partial \boldsymbol{F}} = -\frac{1}{\det(\boldsymbol{F})}\, \boldsymbol{F}^{\text{-t}} \qquad \frac{\partial \ln(\det(\boldsymbol{F})^{\text{-1}})}{\partial \boldsymbol{F}} = -\boldsymbol{F}^{\text{-t}}$$

$$\frac{\partial \ln(\det(\boldsymbol{F}))}{\partial(\det(\boldsymbol{F}))} = \frac{1}{\det(\boldsymbol{F}))} \qquad \frac{\partial \ln^2(\det(\boldsymbol{F}))}{\partial \boldsymbol{F}} = 2\, \ln(\det(\boldsymbol{F}))\boldsymbol{F}^{\text{-t}}$$

neo hooke'ian elasticity

- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
  $+ \frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\, \ln(\det(\boldsymbol{F}))\,]$
- definition of stress

$$\begin{aligned}\boldsymbol{P}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}}\\ &= \tfrac{1}{2}\,\lambda_0\, 2\, \ln(\det \boldsymbol{F})\boldsymbol{F}^{\text{-t}} + \tfrac{1}{2}\mu_0\, 2\, \boldsymbol{F} - \mu_0 \boldsymbol{F}^{\text{-t}}\\ &= \mu_0\, \boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}\end{aligned}$$

- definition of tangent operator

$$\begin{aligned}\mathsf{A}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}\boldsymbol{F}}\psi_0^{\text{neo}} = \mathrm{D}_{\boldsymbol{F}}\boldsymbol{P}^{\text{neo}}\\ &= \lambda_0 \boldsymbol{F}^{\text{-t}} \otimes \boldsymbol{F}^{\text{-t}} + \mu_0 \boldsymbol{I}\overline{\otimes}\boldsymbol{I}\\ &+[\mu_0 - \lambda_0 \ln(\det(\boldsymbol{F}))]\, \boldsymbol{F}^{\text{-t}}\underline{\otimes}\boldsymbol{F}^{\text{-1}}\end{aligned}$$

neo hooke'ian elasticity

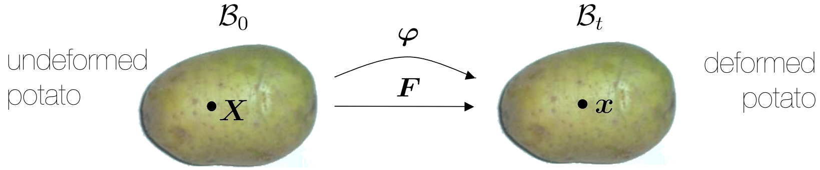
- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(F_{ij}))$
  $+ \frac{1}{2}\,\mu_0[\,F_{ij}F_{ij} - n^{\text{dim}} - 2\, \ln(\det(F_{ij}))\,]$
- definition of stress

$$\begin{aligned}P_{ij}^{\text{neo}} &= \mathrm{D}_{F_{ij}}\psi_0^{\text{neo}}\\ &= \tfrac{1}{2}\,\lambda_0\, 2\, \ln(\det F_{ij})F_{ji}^{\text{-1}} + \tfrac{1}{2}\mu_0\, 2\, F_{ij} - \mu_0 F_{ji}^{\text{-1}}\\ &= \mu_0\, F_{ij} + [\,\lambda_0 \ln(\det(F_{ij})) - \mu_0\,]F_{ji}^{\text{-1}}\end{aligned}$$

- definition of tangent operator

$$\begin{aligned}\mathsf{A}_{ijkl}^{\text{neo}} &= \mathrm{D}_{F_{ij}F_{kl}}\psi_0^{\text{neo}} = \mathrm{D}_{F_{kl}}P_{ij}^{\text{neo}}\\ &= \lambda_0 F_{ji}^{\text{-1}}\; F_{lk}^{\text{-1}} + \mu_0 I_{ik}\; I_{jl}\\ &+[\mu_0 - \lambda_0 \ln(\det(F_{ij}))]\, F_{li}^{\text{-1}}\; F_{jk}^{\text{-1}}\end{aligned}$$

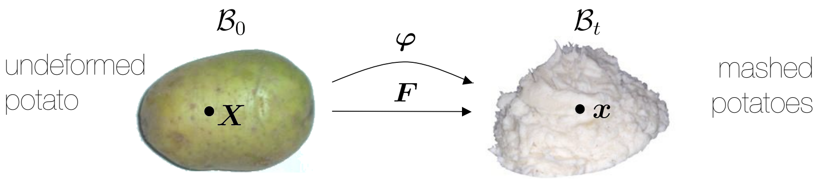
neo hooke'ian elasticity



- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F}))$
  $+ \frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}} \cdot \boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\, \ln(\det(\boldsymbol{F}))\,]$
- definition of stress

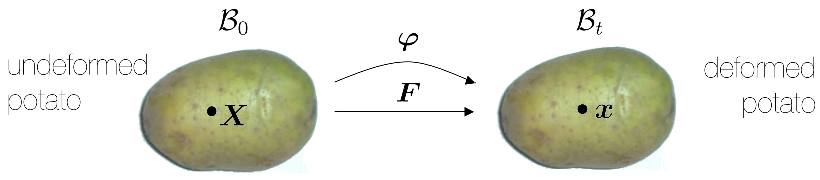
$$\begin{aligned}\boldsymbol{P}^{\text{neo}} &= \mathrm{D}_{\boldsymbol{F}}\psi_0^{\text{neo}}\\ &= \mu_0\, \boldsymbol{F} + [\,\lambda_0 \ln(\det(\boldsymbol{F})) - \mu_0\,]\boldsymbol{F}^{\text{-t}}\end{aligned}$$

neo hooke'ian elasticity



- free energy $\cancel{\psi^{\text{neo}} = \frac{1}{2}\,\lambda\,\ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\,\mu[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]}$
- definition of stress
$$\cancel{\boldsymbol{P}^{\text{neo}} = \rho_0 \mathrm{D}_{\boldsymbol{F}}\psi} = \mu\,\boldsymbol{F} + [\,\lambda \ln(\det(\boldsymbol{F})) - \mu\,]\,\boldsymbol{F}^{\text{-t}}$$
- remember! mashing potatoes is not an elastic process!

neo hooke'ian elasticity

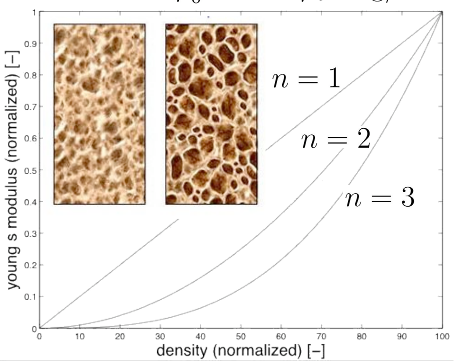


- free energy $\psi_0^{\text{neo}} = \frac{1}{2}\,\lambda_0 \ln^2(\det(\boldsymbol{F})) + \frac{1}{2}\,\mu_0[\,\boldsymbol{F}^{\text{t}}\cdot\boldsymbol{F} : \boldsymbol{I} - n^{\text{dim}} - 2\,\ln(\det(\boldsymbol{F}))\,]$
- large strain - lamé parameters and bulk modulus
$$\lambda = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} \qquad \mu = \frac{E}{2\,[1+\nu]} \qquad \kappa = \frac{E}{3\,[1-2\,\nu]}$$
- small strain - young's modulus and poisson's ratio
$$E = 3\,\kappa\,[1-2\,\nu] \qquad \nu = \frac{3\,\kappa - 2\,\mu}{2\,[3\,\kappa+\mu]}$$

neo hooke'ian elasticity in cellular materials

free energy $\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \psi_0^{\text{neo}}(\boldsymbol{F})$

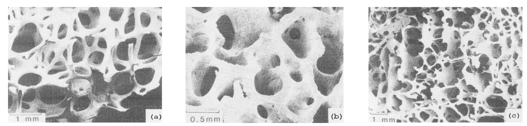
$\mathrm{E} = 3.790\;\rho_0^3\;\mathrm{MPa} \quad \rho_0$ in g/cm^3



Carter & Hayes [1977]

density growth at constant volume

- free energy $\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \psi_0^{\text{neo}}(\boldsymbol{F})$
- stress $\boldsymbol{P} = \left[\frac{\rho_0}{\rho_0^*}\right]^n \boldsymbol{P}^{\text{neo}}(\boldsymbol{F})$
- mass flux $\boldsymbol{R} = R_0\,\nabla_X \rho_0$
- mass source $\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0(\boldsymbol{F}) - \psi_0^*$



constitutive coupling of growth and deformation

Gibson & Ashby [1999]

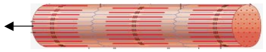
density growth - mass source

$\mathrm{D}_t \rho_0 = \mathcal{R}_0$

1d model problem

$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0 - \psi_0^*$

$l = 1$

f, u  f, u

$n = 2$
$m = 3$

$E = 1 \quad \nu = 0 \quad \rho_0^* = 1 \quad \psi_0^* = 1$

gradually increased workout

$0 < t \le 2 : f = 0.5\,\mathrm{N}$
$2 < t \le 4 : f = 1.0\,\mathrm{N}$
$4 < t \le 6 : f = 1.5\,\mathrm{N}$
$6 < t \le 8 : f = 2.0\,\mathrm{N}$

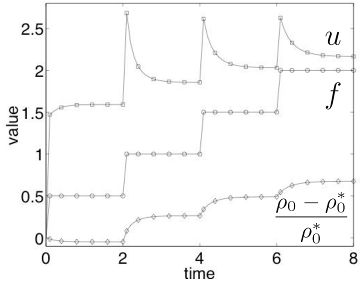
how does the mass source control growth?

density growth - mass source

$\mathrm{D}_t \rho_0 = \mathcal{R}_0$

f  f

$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0 - \psi_0^*$



$f = 0.5\,\mathrm{N} \quad u = 1.5910\,l$
$f = 1.0\,\mathrm{N} \quad u = 1.8559\,l$
$f = 1.5\,\mathrm{N} \quad u = 2.0310\,l$
$f = 2.0\,\mathrm{N} \quad u = 2.1652\,l$

resorption $-1 < \frac{\rho_0 - \rho_0^*}{\rho_0^*} < 0$

growth $0 < \frac{\rho_0 - \rho_0^*}{\rho_0^*} < \infty$

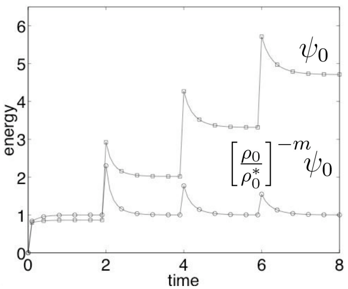
increasing forces causes density increase

density growth - mass source

$\mathrm{D}_t \rho_0 = \mathcal{R}_0$

f  f

$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0 - \psi_0^*$



first time interval $\quad f = 0.5\mathrm{N}$

$u = 1.5910 l \quad \varphi = X + u = 2.5910 l$

$F = \frac{\partial \varphi}{\partial X} = \frac{\partial\, 2.5910\, l}{\partial l} = 2.5910$ homog.

$E = 1 \quad \lambda_0 = \frac{E\,\nu}{[1+\nu][1-2\,\nu]} = 0$

$\nu = 0 \quad \mu_0 = \frac{E}{2\,[1+\nu]} = 0.5$

$\psi_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^n \frac{1}{2}\,\mu_0 [\, F^2 - 1 - 2\,\ln(F)\,]$

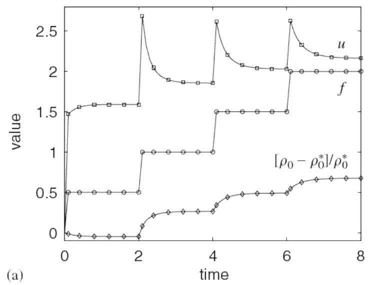
increasing force causes energy increase

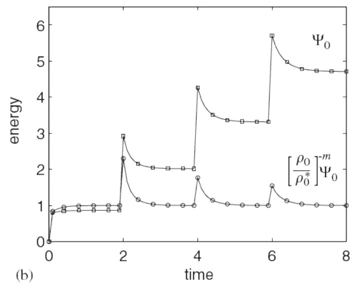
density growth - mass source

$\mathrm{D}_t \rho_0 = \mathcal{R}_0$

f  f

$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m} \psi_0 - \psi_0^*$



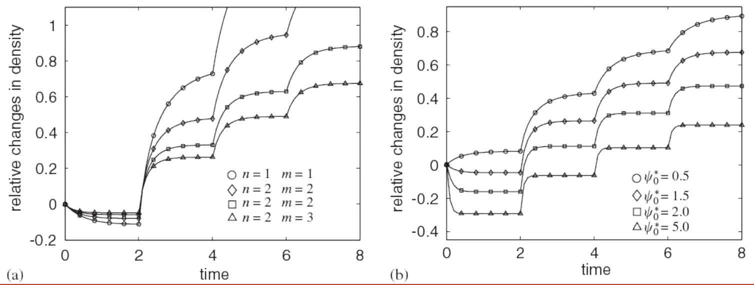


convergence towards biological equilibrium $\mathrm{D}_t \rho_0 = 0$

density growth - mass source

$D_t\rho_0 = \mathcal{R}_0$

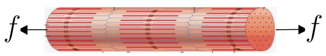
$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0 - \psi_0^*$



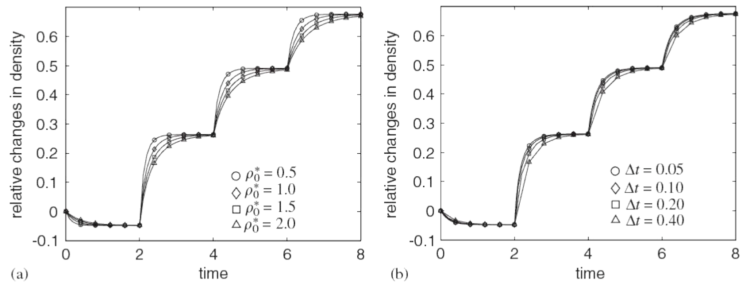
parameter sensitivity wrt n, m, ψ_0^*

density growth - mass source

$D_t\rho_0 = \mathcal{R}_0$



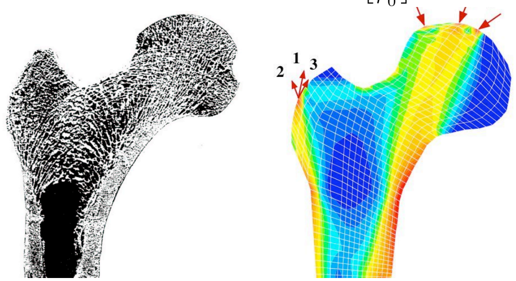
$\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0 - \psi_0^*$



parameter insensitivity wrt $\rho_0^*, \Delta t$

density growth - mass source

$D_t\rho_0 = \mathcal{R}_0$ $\qquad$ $\mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0 - \psi_0^*$

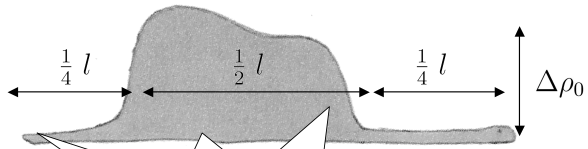


the density develops such that the tissue can just support the given mechanical load

density growth - mass flux

$D_t\rho_0 = \mathrm{Div}(\boldsymbol{R}) \qquad \boldsymbol{R} = R_0 \nabla_X \rho_0$

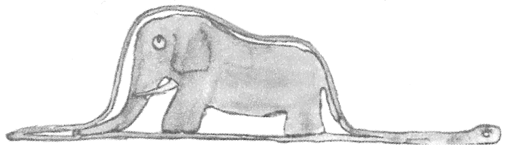
initial hat type density distriubtion



density growth - mass flux

$$\mathrm{D}_t\rho_0 = \mathrm{Div}(\boldsymbol{R}) \qquad \boldsymbol{R} = R_0\,\nabla_X\rho_0$$

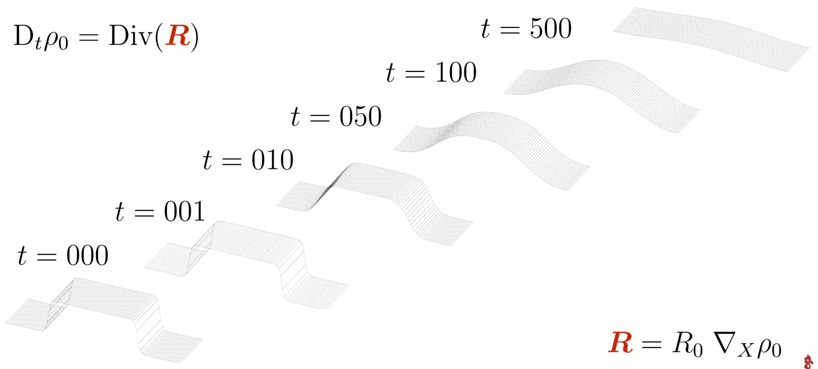
initial hat type density distriubtion



„…mon dissin ne représentait pas un chapeau. il représentait un serpent boa qui digérait un éléphant. j'ai alors dessiné l'intérieur du serpent boa, afin que les grandes personnes puissent comprendre. elles ont toujours besoin d'explications…" Antoine de Saint-Exupéry, Le Petit Prince [1943]

density growth - mass flux

$$\mathrm{D}_t\rho_0 = \mathrm{Div}(\boldsymbol{R})$$

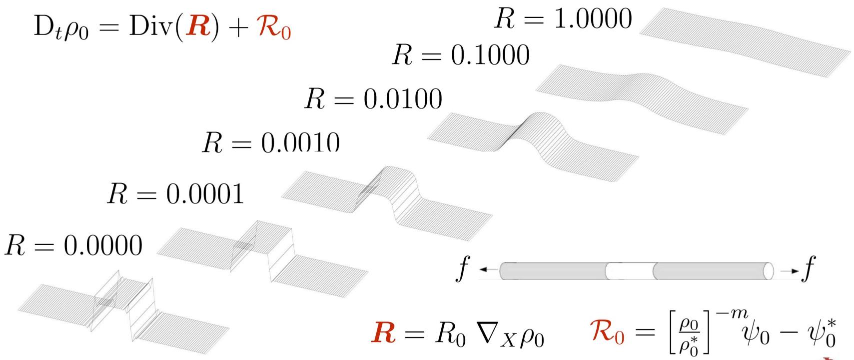


$$\boldsymbol{R} = R_0\,\nabla_X\rho_0$$

equilibration of concentrations

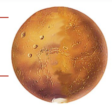
density growth - mass flux & source

$$\mathrm{D}_t\rho_0 = \mathrm{Div}(\boldsymbol{R}) + \mathcal{R}_0$$



$$\boldsymbol{R} = R_0\,\nabla_X\rho_0 \qquad \mathcal{R}_0 = \left[\frac{\rho_0}{\rho_0^*}\right]^{-m}\psi_0 - \psi_0^*$$

smoothing influence of mass flux



density growth - bone loss in space

human spaceflight to mars could become a reality within the next 25 years, but not until some physiological problems are resolved, including an alarming loss of bone mass, fitness and muscle strength. gravity at mars' surface is about 38 percent of that on earth. with lower gravitational forces, bones decrease in mass and density. the rate at which we lose bone in space is 10-15 times greater than that of a post-menopausal woman and there is no evidence that bone loss ever slows in space. further, it is not clear that space travelers will regain that bone on returning to gravity. druing a trip to mars, lasting between 13 and 30 months, unchecked bone loss could make an astronaut's skeleton the equivalent of a 100-year-old person.

http://www.acsm.org

density growth - bone loss in space

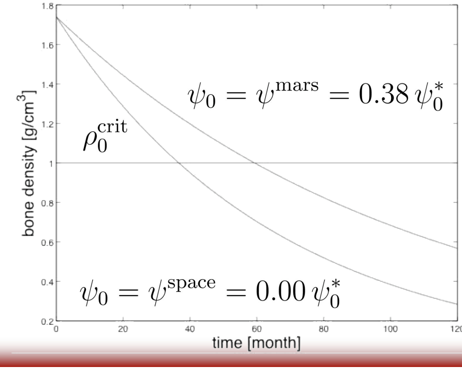
$$\mathrm{D}_t\rho_0 = \mathcal{R}_0 \qquad \mathcal{R}_0 = c\,\frac{\rho_0}{\psi_0^*}\,[\,\psi_0 - \psi_0^*\,]$$

nasa has collected data that humans in space lose bone mass at a rate of $c = 1.5\%/\mathrm{month}$ so far, no astronauts have been in space for more than 14 months but the predicted rate of bone loss seems constant in time. this could be a severe problem if we want to send astronauts on a 3 year trip to mars and back. how long could an astronaut survive in a zero-g environment if we assume the critical bone density to be $\rho_0^{\mathrm{crit}} = 1.00\frac{\mathrm{g}}{\mathrm{cm}^3}$? you can assume an initial density of $\rho_0^* = 1.79\frac{\mathrm{g}}{\mathrm{cm}^3}$!

density growth - bone loss in space

$$\mathrm{D}_t\rho_0 = c\,\rho_0[\,\frac{\psi_0}{\psi_0^*} - 1\,] \qquad \mathrm{D}_t\rho_0 = \frac{1}{\Delta t}[\rho_0^{\mathrm{n+1}} - \rho_0^{\mathrm{n}}]$$

$$\rho_0^{\mathrm{n+1}} = \rho_0^{\mathrm{n}} + c\,\rho_0^{\mathrm{n}}[\,\frac{\psi_0}{\psi_0^*} - 1\,]\,\Delta t \qquad \rho_0(t_0) = 1.79\frac{\mathrm{g}}{\mathrm{cm}^3}$$

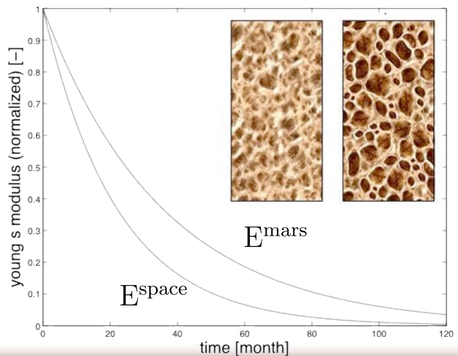


$$\rho_0(36) = 1.0098$$
$$\rho_0(37) = 0.9947$$

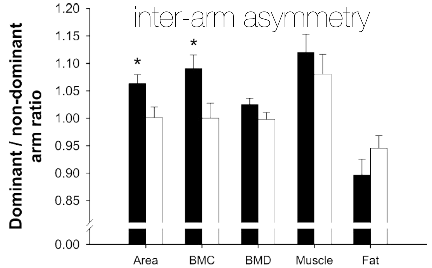
density growth - bone loss in space

$$\mathrm{E} = 3.790\;\rho_0^3\;\mathrm{MPa} \quad \text{with } \rho_0 \text{ in g/cm}^3$$

Carter & Hayes [1977]



density growth - asymmetric bone mass



postmenopausal tennis players,who started their participation in tennis after menarche show greater bone size and mass in the loaded arm. the degree of inter-arm asymmetry in bone mass and bone size is proportional to the length of tennis participation.

Sanchis-Moysi et al. [2004]