04 - kinematic equations - large deformations and growth

continuum mechancis

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is a branch of physics (specifically mechanics) that deals with continuous matter. the fact that matter is made of atoms and that it commonly has some sort of heterogeneous microstructure is ignored in the simplifying approximation that physical quantities, such as energy and momentum, can be handled in the infinitesimal limit. differential equations can thus be employed in solving problems in continuum mechanics.

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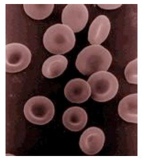
continuum mechanics

continuum mechanics [kənˈtɪn.ju.əm məˈkæn.ɪks] is the branch of mechanics concerned with the stress in solids, liquids and gases and the deformation or flow of these materials. the adjective continuous refers to the simplifying concept underlying the analysis: we disregard the molecular structure of matter and picture it as being without gaps or empty spaces. we suppose that all the mathematical functions entering the theory are continuous functions. this hypothetical continuous material we call a continuum.

Malvern „Introduction to the mechanics of a continuous medium" [1969]

continuum mechanics

continuum hypothesis [kənˈtɪn.ju.əm haɪˈpɑːθ.ə.sɪs] we assume that the characteristic length scale of the microstructure is much smaller than the characteristic length scale of the overall problem, such that the properties at each point can be understood as averages over a characteristic length scale



$$l^{\text{micro}} \ll l^{\text{averg}} \ll l^{\text{conti}}$$

example: biomechanics

$$l^{\text{micro}} = l^{\text{cells}} \approx 10\mu\text{m}$$
$$l^{\text{conti}} = l^{\text{tissue}} \approx 10\text{cm}$$

the continuum hypothesis can be applied when analyzing tissues

the potato equations

- kinematic equations - what's strain? $\epsilon = \frac{\Delta l}{l}$
  general equations that characterize the deformation of a physical body without studying its physical cause
- balance equations - what's stress? $\sigma = \frac{F}{A}$
  general equations that characterize the cause of motion of any body
- constitutive equations - how are they related? $\sigma = E\,\epsilon$
  material specific equations that complement the set of governing equations

the potato equations

- kinematic equations - why not $\epsilon = \frac{\Delta l}{l}$?
  inhomogeneous deformation » non-constant
  finite deformation » non-linear $\boldsymbol{F} = \nabla_X \boldsymbol{\varphi}$
  inelastic deformation » growth tensor $\boldsymbol{F} = \boldsymbol{F}_\mathrm{e} \cdot \boldsymbol{F}_\mathrm{g}$
- balance equations - why not $\sigma = \frac{F}{A}$? $\mathrm{Div}(\boldsymbol{P}) + \rho\,\boldsymbol{b}_0 = \boldsymbol{0}$
  equilibrium in deformed configuration » multiple stress measures
- constitutive equations - why not $\sigma = E\,\epsilon$?
  finite deformation » non-linear $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{F})$
  inelastic deformation » internal variables $\boldsymbol{P} = \boldsymbol{P}(\rho, \boldsymbol{F}, \boldsymbol{F}_\mathrm{g})$

kinematic equations

kinematic equations [kɪnəˈmætɪk ɪˈkweɪ.ʒəns] describe the motion of objects without the consideration of the masses or forces that bring about the motion. the basis of kinematics is the choice of coordinates. the 1st and 2nd time derivatives of the position coordinates give the velocities and accelerations. the difference in placement between the beginning and the final state of two points in a body expresses the numerical value of strain. strain expresses itself as a change in size and/or shape.

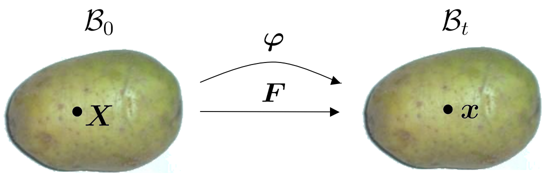
Wikipedia The Free Encyclopedia

kinematic equations

kinematics [kɪnəˈmætɪks] is the study of motion per se, regardless of the forces causing it. the primitive concepts concerned are position, time and body, the latter abstracting into mathematical terms intuitive ideas about aggregations of matter capable of motion and deformation.

Chadwick „Continuum mechanics" [1976]

potato - kinematics



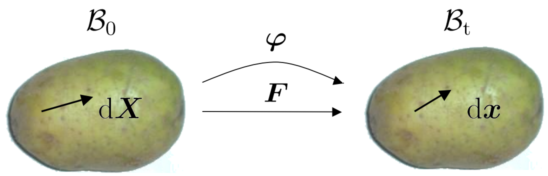
- nonlinear deformation map φ

$$\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t) \quad \text{with} \quad \boldsymbol{\varphi} : \mathcal{B}_0 \times \mathbb{R} \to \mathcal{B}_t$$

- spatial derivative of φ - deformation gradient

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{X} \quad \text{with} \quad \boldsymbol{F} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad \boldsymbol{F} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{X}}\right|_{t \text{ fixed}}$$

potato - kinematics



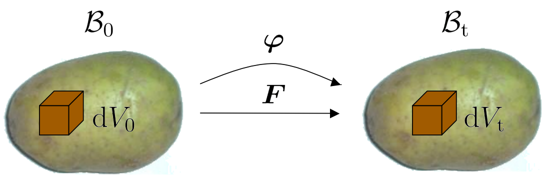
- transformation of line elements - deformation gradient F_{ij}

$$\mathrm{d}x_i = F_{ij}\, \mathrm{d}X_j \quad \text{with} \quad F_{ij} : T\mathcal{B}_0 \to T\mathcal{B}_t \qquad F_{ij} = \left.\frac{\partial \varphi_i}{\partial X_j}\right|_{t \text{ fixed}}$$

- uniaxial tension (incompressible), simple shear, rotation

$$F_{ij}^{\text{uni}} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-\frac{1}{2}} & 0 \\ 0 & 0 & \alpha^{-\frac{1}{2}} \end{bmatrix} \quad F_{ij}^{\text{shr}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad F_{ij}^{\text{rot}} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

potato - kinematics of finite growth



- transformation of volume elements - determinant of $\boldsymbol{F}$

$$\mathrm{d}V_0 = \mathrm{d}\boldsymbol{X}_1 \cdot [\mathrm{d}\boldsymbol{X}_2 \times \mathrm{d}\boldsymbol{X}_3] \qquad \mathrm{d}V_t = \mathrm{d}\boldsymbol{x}_1 \cdot [\mathrm{d}\boldsymbol{x}_2 \times \mathrm{d}\boldsymbol{x}_3]$$

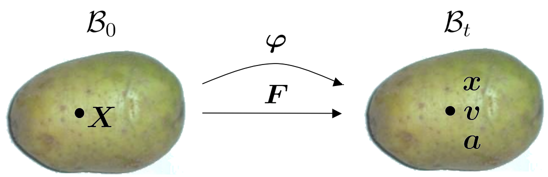
$$= \det([\mathrm{d}\boldsymbol{x}_1, \mathrm{d}\boldsymbol{x}_2, \mathrm{d}\boldsymbol{x}_3])$$

$$= \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3]) \qquad = \det(\boldsymbol{F})\, \det([\mathrm{d}\boldsymbol{X}_1, \mathrm{d}\boldsymbol{X}_2, \mathrm{d}\boldsymbol{X}_3])$$

- changes in volume - determinant of deformation tensor J

$$\mathrm{d}V_t = J\, \mathrm{d}V_0 \qquad J = \det(\boldsymbol{F})$$

potato - kinematics



- temporal derivative of φ - velocity (material time derivative)

$$\boldsymbol{v} = \mathrm{D}_t \boldsymbol{\varphi} = \left.\frac{\partial \boldsymbol{\varphi}}{\partial t}\right|_{X \text{ fixed}} \qquad \text{with} \quad \boldsymbol{v} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

- temporal derivative of $\boldsymbol{v}$ - acceleration

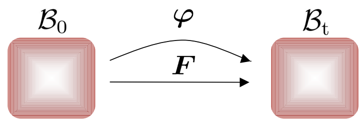
$$\boldsymbol{a} = \mathrm{D}_t \boldsymbol{v} = \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{X \text{ fixed}} = \left.\frac{\partial^2 \boldsymbol{\varphi}}{\partial t^2}\right|_{X \text{ fixed}} \quad \text{with} \quad \boldsymbol{a} : \mathcal{B}_0 \times \mathbb{R} \to \mathbb{R}^3$$

volume growth

volume growth ['val.juːm groʊθ] is conceptually comparable to thermal expansion. in linear elastic problems, growth stresses (such as thermal stresses) can be superposed on the mechanical stress field. in the nonlinear problems considered here, another approach must be used. the fundamental idea is to refer the strain measures in the constitutive equations of each material element to its current zero-stress configuration, which changes as the element grows.

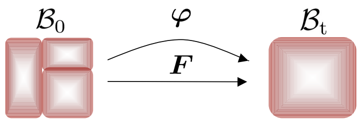
Taber „Biomechanics of growth, remodeling and morphogenesis" [1995]

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

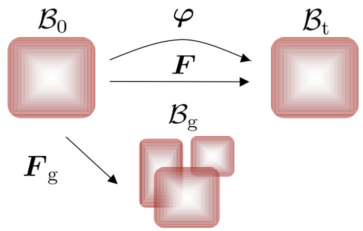
kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

[2] imagine the body is cut into infinitesimal elements each of which is allowed to undergo volumetric growth

kinematics of finite growth



[1] consider an elastic body $\mathcal{B}_0$ at time t_0, unloaded & stressfree

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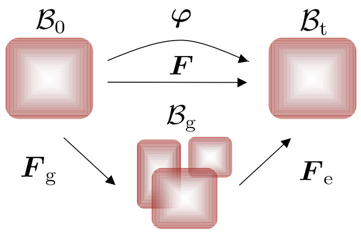
[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

kinematics of finite growth



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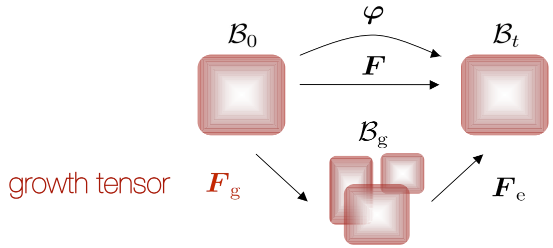
[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

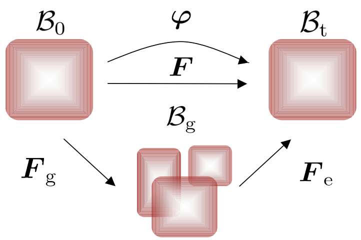
$$\boldsymbol{F} = \boldsymbol{F}_e \cdot \boldsymbol{F}_g$$



multiplicative decomposition

Lee [1969], Simo [1992], Rodriguez, Hoger & Mc Culloch [1994], Epstein & Maugin [2000], Humphrey [2002], Ambrosi & Mollica [2002], Himpel, Kuhl, Menzel & Steinmann [2005]

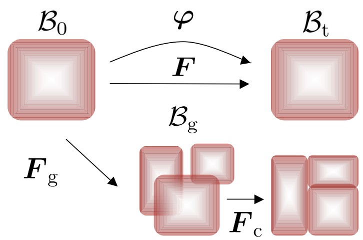
kinematics of finite growth



[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[4] loading generates compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

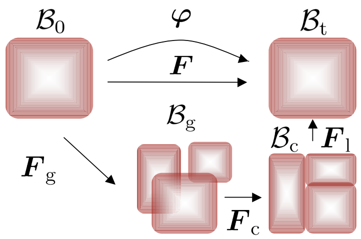


[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

[4] to generate the compatible current configuration $\mathcal{B}_t$

kinematics of finite growth



[3] after growing the elements, $\mathcal{B}_g$ may be incompatible

[3a] we then first apply a deformation $\boldsymbol{F}_c$ to squeeze the elements back together to the compatible configuration $\mathcal{B}_c$

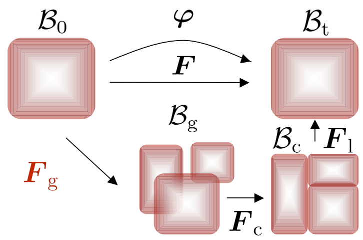
[3b] and then load the compatible configuration $\mathcal{B}_c$ by $\boldsymbol{F}_l$

[4] to generate the compatible current configuration $\mathcal{B}_t$

kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_l \cdot \boldsymbol{F}_c \cdot \boldsymbol{F}_g$$

growth tensor $\boldsymbol{F}_g$



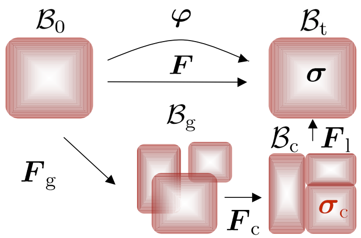
$$\boldsymbol{F}_e = \boldsymbol{F}_l \cdot \boldsymbol{F}_c$$

multiplicative decomposition

Lee [1969], Simo [1992], Rodriguez, Hoger & Mc Culloch [1994], Epstein & Maugin [2000], Humphrey [2002], Ambrosi & Mollica [2002], Himpel, Kuhl, Menzel & Steinmann [2005]

kinematics of finite growth

$$\boldsymbol{F} = \boldsymbol{F}_l \cdot \boldsymbol{F}_c \cdot \boldsymbol{F}_g$$



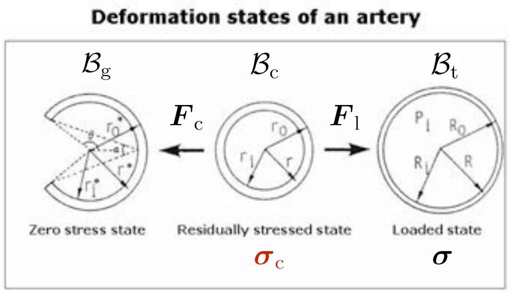
total stress

residual stress

residual stress

the additional deformation of squeezing the grown parts back to a compatible configuration gives rise to residual stresses (see thermal stresses)

kinematics of finite growth



residual stress

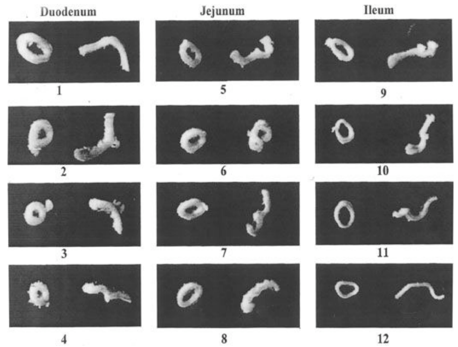
Horný, Chlup, Žitný, Mackov [2006]

the opening angle experiment

„an existence of residual strains in human arteries is well known. It can be observed as an opening up of a circular arterial segment after a radial cut. an opening angle of the arterial segment is used as a measure of the residual strains generally."

Fung [1990], Horný, Chlup, Žitný, Mackov [2006]

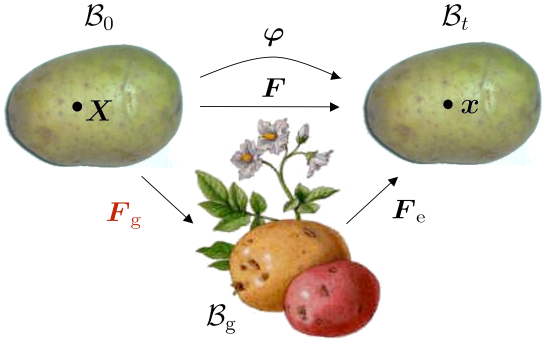
the opening angle experiment



„photographs showing specimens obtained from different locations in the intestine in the no-load state (left, closed rings) and the zero-stress state (right, open sectors). the rings of jejunum (site 5 to site 8) turned inside out when cut open"

Zhao, Sha, Zhuang & Gregersen [2002]

potato - kinematics of finite growth



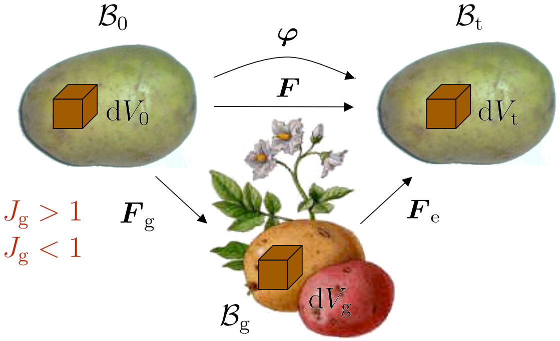
- incompatible growth configuration $\mathcal{B}_{\rm g}$ & growth tensor $\boldsymbol{F}_{\rm g}$

$$\boldsymbol{F} = \boldsymbol{F}_{\rm e} \cdot \boldsymbol{F}_{\rm g}$$

Rodriguez, Hoger & McCulloch [1994]

potato - kinematics of finite growth

growth $J_{\rm g} > 1$
resorption $J_{\rm g} < 1$



- changes in volume - determinant of growth tensor $J_{\rm g}$

$$\mathrm{d}V_{\rm g} = J_{\rm g}\,\mathrm{d}V_0 \qquad J_{\rm g} = \det(\boldsymbol{F}_{\rm g})$$