CONTINUUM MECHANICS

lecture notes 2003

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1.1.1.1 Notation

• Einstein's summation convention

$$u_i = \sum_{j=1}^{3} A_{ij}\, x_j + b_i = A_{ij}\, x_j + b_i \tag{1.1.1}$$

summation over indices that appear twice in a term or symbol, with silent (dummy) index j and free index i, and thus

$$\begin{aligned} u_1 &= A_{11}\, x_1 + A_{12}\, x_2 + A_{13}\, x_3 + b_1 \\ u_2 &= A_{21}\, x_1 + A_{22}\, x_2 + A_{23}\, x_3 + b_2 \\ u_3 &= A_{31}\, x_1 + A_{32}\, x_2 + A_{33}\, x_3 + b_3 \end{aligned} \tag{1.1.2}$$

• Kronecker symbol δ_{ij}

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{1.1.3}$$

multiplication with Kronecker symbol corresponds to exchange of silent index with free index of Kronecker symbol

$$u_i = \delta_{ij}\, u_j \tag{1.1.4}$$

• permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{e}_{ijk} = \begin{cases} 1 & \text{for} \quad \{i, j, k\} \quad \text{... even permutation} \\ -1 & \text{for} \quad \{i, j, k\} \quad \text{... odd permutation} \\ 0 & \text{for} \quad \quad \quad \quad \text{... else} \end{cases} \tag{1.1.5}$$

1.1.1.2 Euclidian vector space

• consider linear vector space $\mathcal{V}^3$ characterized through addition of its elements $\boldsymbol{u}, \boldsymbol{v}$ and multiplication with real scalars α, β

$$\begin{array}{lll} \alpha, \beta \in \mathcal{R} & \mathcal{R} & \text{... real numbers} \\ \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}^3 & \mathcal{V}^3 & \text{... linear vector space} \end{array}$$

definition of linear vector space $\mathcal{V}^3$ through the following axioms

$$\begin{aligned} \alpha\,(\boldsymbol{u} + \boldsymbol{v}) &= \alpha\,\boldsymbol{u} + \alpha\,\boldsymbol{v} \\ (\alpha + \beta)\,\boldsymbol{u} &= \alpha\,\boldsymbol{u} + \beta\,\boldsymbol{u} \\ (\alpha\,\beta)\,\boldsymbol{u} &= \alpha\,(\beta\,\boldsymbol{u}) \end{aligned} \tag{1.1.6}$$

zero element and identity

$$0\,\boldsymbol{u} = \boldsymbol{0} \qquad 1\,\boldsymbol{u} = \boldsymbol{u} \tag{1.1.7}$$

linear independence of elements $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 \in \mathcal{V}^3$ if $\alpha_1 = \alpha_2 = \alpha_3 = 0$ is the only (trivial) solution to

$$\alpha_i\,\boldsymbol{e}_i = 0 \tag{1.1.8}$$

• consider linear vector space $\mathcal{V}^3$ equipped with a norm $n(\boldsymbol{u})$ mapping elements of the linear vector space $\mathcal{V}^3$ to the space of real numbers $\mathcal{R}$

$$n : \; \mathcal{V}^3 \rightarrow \mathcal{R} \qquad \text{norm} \tag{1.1.9}$$

definition of norm through the following axioms

$$\begin{aligned} & n(\boldsymbol{u}) \geq 0 \qquad\qquad n(\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0} \\ & n(\alpha\, \boldsymbol{u}) = |\alpha|\, n(\boldsymbol{u}) \\ & n(\boldsymbol{u} + \boldsymbol{v}) \leq n(\boldsymbol{u}) + n(\boldsymbol{v}) \\ & n^2(\boldsymbol{u} + \boldsymbol{v}) + n^2(\boldsymbol{u} - \boldsymbol{v}) = 2\,[n^2(\boldsymbol{u}) + n^2(\boldsymbol{v})] \end{aligned} \tag{1.1.10}$$

• consider Euclidian vector space $\mathcal{E}^3$ equipped with the Euclidian norm

$$n(\boldsymbol{u}) = ||\boldsymbol{u}|| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = [u_1^2 + u_2^2 + u_3^2]^{1/2} \tag{1.1.11}$$

mapping elements of the Euclidian vector space $\mathcal{E}^3$ to the space of real numbers $\mathcal{R}$

$$n : \; \mathcal{E}^3 \rightarrow \mathcal{R} \qquad \text{Euclidian norm} \tag{1.1.12}$$

representation of three–dimensional vector $\boldsymbol{a} \in \mathcal{E}^3$

$$\boldsymbol{a} = a_i\, \boldsymbol{e}_i = a_1\, \boldsymbol{e}_1 + a_2\, \boldsymbol{e}_2 + a_3\, \boldsymbol{e}_3 \tag{1.1.13}$$

with a_1, a_2, a_3 coordinates (components) of $\boldsymbol{a}$ relative to the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

$$\boldsymbol{a} = [\, a_1, a_2, a_3\,]^{\mathrm{t}} \tag{1.1.14}$$

1.1.1.3 Scalar product

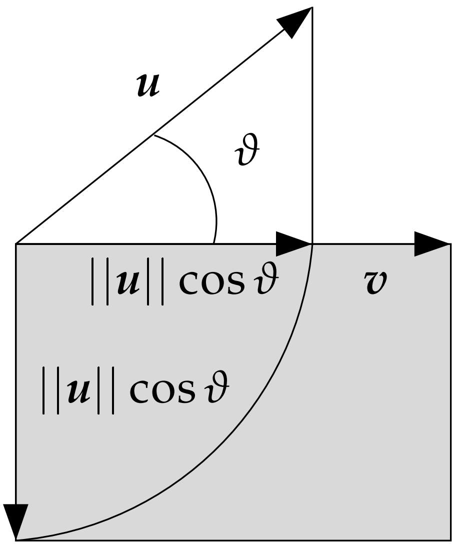
Euclidian norm enables the definition of scalar (inner) product between two vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and introduces a scalar $\alpha \in \mathcal{R}$

$$\boldsymbol{u} \cdot \boldsymbol{v} = \alpha \tag{1.1.15}$$

geometric interpretation with $0 \leq \vartheta \leq \pi$ being the angle enclosed by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then $||\boldsymbol{u}|| \cos\vartheta$ can be interpreted as the projection of $\boldsymbol{u}$ onto the direction of $\boldsymbol{v}$ and

$$\boldsymbol{u} \cdot \boldsymbol{v} = ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \, \cos\vartheta$$

corresponds to the grey area in the picture



with the above interpretation with $0 \leq \vartheta \leq \pi$, obviously

$$||\boldsymbol{u} \cdot \boldsymbol{v}|| \leq ||\boldsymbol{u}|| \, ||\boldsymbol{v}|| \tag{1.1.16}$$

properties of scalar product

$$\begin{aligned} \boldsymbol{u} \cdot \boldsymbol{v} &= \boldsymbol{v} \cdot \boldsymbol{u} \\ (\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) \cdot \boldsymbol{w} &= \alpha\,(\boldsymbol{u} \cdot \boldsymbol{w}) + \beta\,(\boldsymbol{v} \cdot \boldsymbol{w}) \\ \boldsymbol{w} \cdot (\alpha\,\boldsymbol{u} + \beta\,\boldsymbol{v}) &= \alpha\,(\boldsymbol{w} \cdot \boldsymbol{u}) + \beta\,(\boldsymbol{w} \cdot \boldsymbol{v}) \end{aligned} \tag{1.1.17}$$

positive definiteness of scalar product

$$\boldsymbol{u} \cdot \boldsymbol{u} \geq 0, \quad \boldsymbol{u} \cdot \boldsymbol{u} = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0} \tag{1.1.18}$$

orthogonal vectors $\boldsymbol{u}$ and $\boldsymbol{v}$

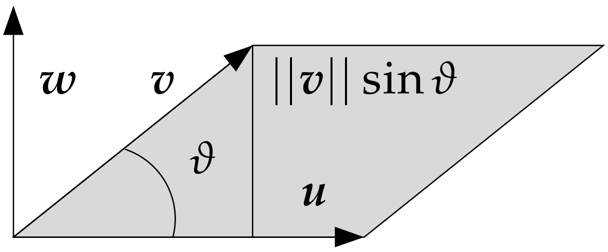
$$\boldsymbol{u} \cdot \boldsymbol{v} = 0 \quad \Leftrightarrow \quad \boldsymbol{u} \perp \boldsymbol{v} \tag{1.1.19}$$

1.1.1.4 Vector product

vector product of two vectors $\boldsymbol{u}, \boldsymbol{v}$ defines a new vector $\boldsymbol{w} \in \mathcal{E}^3$

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{w} \tag{1.1.20}$$

geometric interpretation with $0 \leq \vartheta \leq \pi$ being the angle enclosed by the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then $||\boldsymbol{v}||\ \sin\vartheta$ can be interpreted as the height of the grey polygon and



$$\boldsymbol{u} \times \boldsymbol{v} = ||\boldsymbol{u}||\ ||\boldsymbol{v}||\ \sin\vartheta\, \boldsymbol{n}$$

introduces the vector $\boldsymbol{w}$ orthogonal to $\boldsymbol{u}$ and $\boldsymbol{v}$ whereby its length corresponds to the grey area

with the above interpretation, obviously $\boldsymbol{u}$ parallel to $\boldsymbol{v}$ if

$$\boldsymbol{u} \times \boldsymbol{v} = \boldsymbol{0} \qquad \Leftrightarrow \qquad \boldsymbol{u} \parallel \boldsymbol{v} \tag{1.1.21}$$

index representation of $\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v}$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} u_2\, v_3 & - & u_3\, v_2 \\ u_3\, v_1 & - & u_1\, v_3 \\ u_1\, v_2 & - & u_2\, v_1 \end{bmatrix} \tag{1.1.22}$$

properties of vector product

$$\begin{aligned} &\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u} \\ &(\alpha\, \boldsymbol{u} + \beta\, \boldsymbol{v}) \times \boldsymbol{w} = \alpha\, (\boldsymbol{u} \times \boldsymbol{w}) + \beta\, (\boldsymbol{v} \times \boldsymbol{w}) \\ &\boldsymbol{u} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = 0 \\ &(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) = (\boldsymbol{u} \cdot \boldsymbol{u})\, (\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \end{aligned} \tag{1.1.23}$$

1.1.1.5 Scalar triple vector product

scalar triple vector product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ introduces a scalar $\alpha \in \mathcal{R}$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \alpha \tag{1.1.24}$$

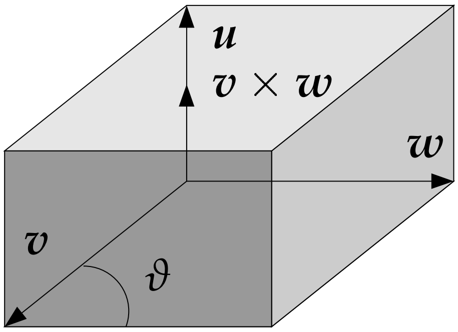
geometric interpretation
with vector product

$$\boldsymbol{v} \times \boldsymbol{w} = ||\boldsymbol{v}|| \, ||\boldsymbol{w}|| \, \sin\vartheta \, \boldsymbol{n}$$

defining aera of ground surface

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$$

defines volume of parallelepiped



obviously

$$\begin{aligned} \alpha &= \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) \\ -\alpha &= \boldsymbol{u} \cdot (\boldsymbol{w} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w}) = \boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u}) \end{aligned} \tag{1.1.25}$$

index representation of $\alpha = [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$

$$\alpha = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_3 w_1) \tag{1.1.26}$$

properties of scalar triple product

$$\begin{aligned} [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] &= [\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}] = [\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}] \\ &= -[\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}] = -[\boldsymbol{v}, \boldsymbol{u}, \boldsymbol{w}] = -[\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u}] \\ [\alpha\boldsymbol{u} + \beta\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}] &= \alpha\, [\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{d}] + \beta\, [\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{d}] \end{aligned} \tag{1.1.27}$$

three vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ are linearly independent if

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \neq 0 \tag{1.1.28}$$

1.1.2 Tensor algebra

1.1.2.1 Notation

Second order tensors

tensor (dyadic) product $\boldsymbol{u} \otimes \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ introduces a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} \tag{1.1.29}$$

introducing $\boldsymbol{u} = u_i\, \boldsymbol{e}_i$ and $\boldsymbol{v} = v_j\, \boldsymbol{e}_j$ yields index representation of three-dimensional second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.30}$$

with $A_{ij} = u_i\, v_j$ coordinates (components) of $\boldsymbol{A}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$, matrix representation of coordinates

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1.1.31}$$

transpose of second order tensor $\boldsymbol{A}^{\mathrm{t}}$

$$\boldsymbol{A}^{\mathrm{t}} = (\boldsymbol{u} \otimes \boldsymbol{v})^{\mathrm{t}} = \boldsymbol{v} \otimes \boldsymbol{u} \tag{1.1.32}$$

introducing $\boldsymbol{u} = u_i\, \boldsymbol{e}_i$ and $\boldsymbol{v} = v_j\, \boldsymbol{e}_j$ yields index representation of transpose of second order tensor $\boldsymbol{A}^{\mathrm{t}}$

$$\boldsymbol{A}^{\mathrm{t}} = A_{ji}\, \boldsymbol{e}_j \otimes \boldsymbol{e}_i \tag{1.1.33}$$

with $A_{ji} = v_j\, u_i$ coordinates (components) of $\boldsymbol{A}$ relative to the tensor basis $\boldsymbol{e}_j \otimes \boldsymbol{e}_i$, matrix representation of coordinates

$$[A_{ji}] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \tag{1.1.34}$$

second order unit tensor $\boldsymbol{I}$ in terms of Kronecker symbol δ_{ij}

$$\boldsymbol{I} = \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.1.35}$$

matrix representation of coordinates δ_{ij}

$$[\delta_{ji}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.1.36}$$

Third order tensors

tensor (dyadic) product $\boldsymbol{A} \otimes \boldsymbol{v}$ of second order tensor $\boldsymbol{A}$ and vectors $\boldsymbol{u}$ introduces a third order tensor $\overset{3}{\boldsymbol{a}}$

$$\overset{3}{\boldsymbol{a}} = \boldsymbol{A} \otimes \boldsymbol{v} \tag{1.1.37}$$

introducing $\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{u} = u_k\, \boldsymbol{e}_k$ yields index representation of three-dimensional third order tensor $\overset{3}{\boldsymbol{a}}$

$$\overset{3}{\boldsymbol{a}} = \overset{3}{a}_{ijk}\;\; \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{1.1.38}$$

with $\overset{3}{a}_{ijk} = \; A_{ij}\, u_k$ coordinates (components) of $\overset{3}{\boldsymbol{a}}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$

third order permutation tensor $\overset{3}{\boldsymbol{e}}$ in terms of permutation symbol $\overset{3}{e}_{ijk}$

$$\overset{3}{\boldsymbol{e}} = \overset{3}{e}_{ijk}\;\; \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \tag{1.1.39}$$

Fourth order tensors

tensor (dyadic) product $\boldsymbol{A} \otimes \boldsymbol{B}$ of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ introduces a fourth order tensor $\mathbb{A}$

$$\mathbb{A} = \boldsymbol{A} \otimes \boldsymbol{B} \tag{1.1.40}$$

introducing $\boldsymbol{A} = A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and $\boldsymbol{B} = B_{kl}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ yields index representation of three-dimensional fourth order tensor $\mathbb{A}$

$$\mathbb{A} = A_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.41}$$

with $A_{ijkl} = A_{ij}\, B_{kl}$ coordinates (components) of $\mathbb{A}$ relative to the tensor basis $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$

fourth order unit tensor $\mathbb{I}$

$$\mathbb{I} = \delta_{ik}\, \delta_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.42}$$

transpose fourth order unit tensor $\mathbb{I}^{\mathrm{t}}$

$$\mathbb{I}^{\mathrm{t}} = \delta_{il}\, \delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.43}$$

symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{sym}}$

$$\mathbb{I}^{\mathrm{sym}} = \tfrac{1}{2}\, [\delta_{ik}\, \delta_{jl} + \delta_{il}\, \delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.44}$$

skew-symmetric fourth order unit tensor $\mathbb{I}^{\mathrm{skw}}$

$$\mathbb{I}^{\mathrm{skw}} = \tfrac{1}{2}\, [\delta_{ik}\, \delta_{jl} - \delta_{il}\, \delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.45}$$

volumetric fourth order unit tensor $\mathbb{I}^{\mathrm{vol}}$

$$\mathbb{I}^{\mathrm{vol}} = \tfrac{1}{3}\, \delta_{ij}\, \delta_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.46}$$

deviatoric fourth order unit tensor $\mathbb{I}^{\mathrm{dev}}$

$$\mathbb{I}^{\mathrm{dev}} = [\tfrac{1}{3}\, \delta_{ij}\, \delta_{kl} + \tfrac{1}{2}\, \delta_{ik}\, \delta_{jl} + \tfrac{1}{2}\, \delta_{il}\, \delta_{jk}]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{1.1.47}$$

1.1.2.2 Scalar products

• scalar product $\boldsymbol{A} \cdot \boldsymbol{u}$ between second order tensor $\boldsymbol{A}$ and vector $\boldsymbol{u}$ defines a new vector $\boldsymbol{v} \in \mathcal{E}^3$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{u} &= (A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (u_k \, \boldsymbol{e}_k) \\ &= A_{ij} \, u_k \, \delta_{jk} \, \boldsymbol{e}_i = A_{ij} \, u_j \, \boldsymbol{e}_i = v_i \boldsymbol{e}_i = \boldsymbol{v} \end{aligned} \tag{1.1.48}$$

second order zero tensor $\boldsymbol{0}$, second order identity tensor $\boldsymbol{I}$

$$\boldsymbol{0} \cdot \boldsymbol{a} = \boldsymbol{0} \qquad \boldsymbol{I} \cdot \boldsymbol{a} = \boldsymbol{a} \tag{1.1.49}$$

positive semi-definiteness of second order tensor $\boldsymbol{A}$

$$\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{a} \geq 0 \tag{1.1.50}$$

positive definiteness of second order tensor $\boldsymbol{A}$

$$\boldsymbol{a} \cdot \boldsymbol{A} \cdot \boldsymbol{a} > 0 \tag{1.1.51}$$

properties of scalar product

$$\begin{aligned} &\boldsymbol{A} \cdot (\alpha \boldsymbol{a} + \beta \boldsymbol{b}) = \alpha \, (\boldsymbol{A} \cdot \boldsymbol{a}) + \beta \, (\boldsymbol{A} \cdot \boldsymbol{b}) \\ &(\boldsymbol{A} + \boldsymbol{B}) \cdot \boldsymbol{a} = \boldsymbol{A} \cdot \boldsymbol{a} + \boldsymbol{B} \cdot \boldsymbol{a} \\ &(\alpha \, \boldsymbol{A}) \cdot \boldsymbol{a} = \alpha \, (\boldsymbol{A} \cdot \boldsymbol{a}) \end{aligned} \tag{1.1.52}$$

• scalar product $\boldsymbol{A} \cdot \boldsymbol{B}$ between two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ defines a second order tensor $\boldsymbol{C}$

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{B} &= (A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl} \, \boldsymbol{e}_k \otimes \boldsymbol{e}_l) \\ &= A_{ij} \, B_{kl} \, \delta_{ik} \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\ &= A_{ij} \, B_{jl} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_l = C_{il} \boldsymbol{e}_i \otimes \boldsymbol{e}_l = \boldsymbol{C} \end{aligned} \tag{1.1.53}$$

second order zero tensor $\boldsymbol{0}$, second order identity tensor $\boldsymbol{I}$

$$\boldsymbol{0} \cdot \boldsymbol{A} = \boldsymbol{0} \qquad \boldsymbol{I} \cdot \boldsymbol{A} = \boldsymbol{A} \tag{1.1.54}$$

properties of scalar product

$$\begin{aligned}
&\alpha\,(\boldsymbol{A}\cdot\boldsymbol{B}) = (\alpha\,\boldsymbol{A})\cdot\boldsymbol{B} = \boldsymbol{A}\cdot(\alpha\,\boldsymbol{B})\\
&\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}\\
&(\boldsymbol{A}+\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot\boldsymbol{C}+\boldsymbol{B}\cdot\boldsymbol{C}
\end{aligned} \tag{1.1.55}$$

properties in terms of transpose $\boldsymbol{A}^{\mathrm{t}}$ of a tensor $\boldsymbol{A}$

$$\begin{aligned}
&\boldsymbol{a}\cdot(\boldsymbol{A}^{\mathrm{t}}\cdot\boldsymbol{b}) = \boldsymbol{b}\cdot(\boldsymbol{A}\cdot\boldsymbol{a})\\
&(\alpha\,\boldsymbol{A}+\beta\,\boldsymbol{B})^{\mathrm{t}} = \alpha\,\boldsymbol{A}^{\mathrm{t}}+\beta\,\boldsymbol{B}^{\mathrm{t}}\\
&(\boldsymbol{A}\cdot\boldsymbol{B})^{\mathrm{t}} = \boldsymbol{B}^{\mathrm{t}}\cdot\boldsymbol{A}^{\mathrm{t}}
\end{aligned} \tag{1.1.56}$$

• scalar product $\boldsymbol{A}:\boldsymbol{B}$ between two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ defines a scalar $\alpha\in\mathcal{R}$

$$\begin{aligned}
\boldsymbol{A}:\boldsymbol{B} &= (A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j):(B_{kl}\,\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\\
&= A_{ij}\,B_{kl}\,\delta_{ik}\,\delta_{jl} = A_{ij}\,B_{ij} = \alpha
\end{aligned} \tag{1.1.57}$$

• scalar product $\mathbb{A}:\boldsymbol{B}$ between fourth order tensor $\mathbb{A}$ and second order tensor $\boldsymbol{B}$ defines a new second order tensor $\boldsymbol{C}$

$$\begin{aligned}
\mathbb{A}:\boldsymbol{B} &= (A_{ijkl}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(B_{mn}\,\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\
&= A_{ijkl}\,B_{mn}\,\delta_{km}\,\delta_{ln}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j\\
&= A_{ijkl}\,B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \boldsymbol{A}
\end{aligned} \tag{1.1.58}$$

1.1.2.3 Dyadic product

• tensor (dyadic) product $\boldsymbol{u} \otimes \boldsymbol{v}$ of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ introduces a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v} = u_i \, \boldsymbol{e}_i \otimes v_j \boldsymbol{e}_j = u_i \, v_j \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = A_{ij} \, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (1.1.59)$$

properties of dyadic product

$$\begin{aligned}
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w}) \, \boldsymbol{u} \\
&(\alpha \, \boldsymbol{u} + \beta \, \boldsymbol{v}) \otimes \boldsymbol{w} = \alpha \, (\boldsymbol{u} \otimes \boldsymbol{w}) + \beta \, (\boldsymbol{v} \otimes \boldsymbol{w}) \\
&\boldsymbol{u} \otimes (\alpha \, \boldsymbol{v} + \beta \, \boldsymbol{w}) = \alpha \, (\boldsymbol{u} \otimes \boldsymbol{v}) + \beta \, (\boldsymbol{u} \otimes \boldsymbol{w}) \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot (\boldsymbol{w} \otimes \boldsymbol{x}) = (\boldsymbol{v} \cdot \boldsymbol{w}) \, (\boldsymbol{u} \otimes \boldsymbol{x}) \\
&\boldsymbol{A} \cdot (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{A} \cdot \boldsymbol{u}) \otimes \boldsymbol{v} \\
&(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{A} = \boldsymbol{u} \otimes (\boldsymbol{A}^{\mathrm{t}} \cdot \boldsymbol{v})
\end{aligned} \quad (1.1.60)$$

or in index notation

$$\begin{aligned}
&(u_i \, v_j) \, w_j = (v_j \, w_j) \, u_i \\
&(\alpha \, u_i + \beta \, v_i) w_j = \alpha \, (u_i \, w_j) + \beta \, (v_i \, w_j) \\
&u_i (\alpha \, v_j + \beta \, w_j) = \alpha \, (u_i \, v_j) + \beta \, (u_i \, w_j) \\
&(u_i \, v_j)(w_j \, x_k) = (v_j \, w_j) \, (u_i x_k) \\
&A_{ij} (u_j \, v_k) = (A_{ij} \, u_i) v_k \\
&(u_i \, v_j) A_{jk} = u_i (A_{kj} \, v_j)
\end{aligned} \quad (1.1.61)$$

1.1.2.4 Scalar triple vector product

consider the set of Cartesian base vectors $\{\boldsymbol{e}_i\}_{i=1,2,3}$ and an arbitrary second set of base vectors $\{\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\}$ with scalar triple product $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]$, with arbitrary second order tensor $\boldsymbol{A}$, evaluate

$$[\,[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{A}\cdot\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{v},\boldsymbol{A}\cdot\boldsymbol{w}]\,]\,/\,[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \quad (1.1.62)$$

with index representation of each term according to

$$[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]=[\boldsymbol{A}\cdot(u_i\,\boldsymbol{e}_i),(v_j\,\boldsymbol{e}_j),(w_k\,\boldsymbol{e}_k)]=u_i\,v_j\,w_k\,[\boldsymbol{A}\cdot\boldsymbol{e}_i,\boldsymbol{e}_j,\boldsymbol{e}_k] \quad (1.1.63)$$

expression (1.1.62) can be rewritten as

$$u_i v_j w_k[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]/[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \quad (1.1.64)$$

term in brackets remains unchanged upon cyclic permutation of $\{\boldsymbol{e}_i\}_{i=1,2,3}$, its sign reverses upon non–cyclic permutations, thus

$$\begin{aligned} u_i v_j w_k \overset{3}{e}_{ijk}\,&[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]/[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \\ &= \pm[[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]] \end{aligned} \quad (1.1.65)$$

the above expression according to (1.1.62) is thus invariant under the choice of base system, it yields the same scalar value I_A for arbitrary base systems

$$\begin{aligned} I_A &= [\,[\boldsymbol{A}\cdot\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{A}\cdot\boldsymbol{v},\boldsymbol{w}]+[\boldsymbol{u},\boldsymbol{v},\boldsymbol{A}\cdot\boldsymbol{w}]\,]/\,[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}] \\ &= [[\boldsymbol{A}\cdot\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{A}\cdot\boldsymbol{e}_2,\boldsymbol{e}_3]+[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{A}\cdot\boldsymbol{e}_3]]/[\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3] \end{aligned} \quad (1.1.66)$$

I_A is called the first invariant of the second order tensor $\boldsymbol{A}$

1.1.2.5 Invariants of second order tensors

the following property of the scalar triple product

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) \tag{1.1.67}$$

introduces three scalar–valued quantities I_A, II_A, III_A associated with the second order tensor $\boldsymbol{A}$

$$\begin{aligned} [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] + [\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{w}] + [\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] &= I_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \\ [\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] + [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] + [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{w}] &= II_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \\ [\boldsymbol{A}\cdot\boldsymbol{u}, \boldsymbol{A}\cdot\boldsymbol{v}, \boldsymbol{A}\cdot\boldsymbol{w}] &= III_A[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] \end{aligned} \tag{1.1.68}$$

the proof of II_A, III_A being invariant for different base systems $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$ is similar to the one for I_A

I_A, II_A, III_A are called the three principal invariants of $\boldsymbol{A}$ which can be expressed as

$$\begin{aligned} I_A &= \mathrm{tr}\,(\boldsymbol{A}) & \partial_A\, I_A &= \boldsymbol{I} \\ II_A &= \tfrac{1}{2}\left[\mathrm{tr}^2\,(\boldsymbol{A}) - \mathrm{tr}\,(\boldsymbol{A}^2)\right] & \partial_A\, II_A &= I_A\,\boldsymbol{I} - \boldsymbol{A} \\ III_A &= \det\,(\boldsymbol{A}) & \partial_A\, III_A &= III_A\,\boldsymbol{A}^{-\mathrm{t}} \end{aligned} \tag{1.1.69}$$

alternatively, we could work with the three basic invariants $\bar{I}_A$, $\bar{II}_A$, $\bar{III}_A$ of $\boldsymbol{A}$ which are more common in the context of anisotropy

$$\begin{aligned} \bar{I}_A &= \boldsymbol{A}^1 : \boldsymbol{I} \\ \bar{II}_A &= \boldsymbol{A}^2 : \boldsymbol{I} \\ \bar{III}_A &= \boldsymbol{A}^3 : \boldsymbol{I} \end{aligned} \tag{1.1.70}$$

1.1.2.6 Trace of second order tensors

trace $\text{tr}\,(\boldsymbol{A})$ of a second order tensor $\boldsymbol{A} = \boldsymbol{u} \otimes \boldsymbol{v}$ introduces a scalar $\text{tr}\,(\boldsymbol{A}) \in \mathcal{R}$

$$\text{tr}\,(\boldsymbol{u} \otimes \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{v} \tag{1.1.71}$$

such that $\text{tr}\,(\boldsymbol{A})$ is the sum of the diagonal entries A_{ii} of $\boldsymbol{A}$

$$\begin{aligned} \text{tr}(\boldsymbol{A}) &= \text{tr}(A_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \\ &= A_{ij}\, \text{tr}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) = A_{ij}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j \\ &= A_{ij}\, \delta_{ij} = A_{ii} = A_{11} + A_{22} + A_{33} \end{aligned} \tag{1.1.72}$$

with

$$I_A = \bar{I}_A = \text{tr}(\boldsymbol{A}) \tag{1.1.73}$$

properties of the trace of second order tensors

$$\begin{aligned} &\text{tr}\,(\boldsymbol{I}) = 3 \\ &\text{tr}\,(\boldsymbol{A}^{\text{t}}) = \text{tr}\,(\boldsymbol{A}) \\ &\text{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}) = \text{tr}\,(\boldsymbol{B} \cdot \boldsymbol{A}) \\ &\text{tr}\,(\alpha\, \boldsymbol{A} + \beta\, \boldsymbol{B}) = \alpha\, \text{tr}(\boldsymbol{A}) + \beta\, \text{tr}(\boldsymbol{B}) \\ &\text{tr}\,(\boldsymbol{A} \cdot \boldsymbol{B}^{\text{t}}) = \boldsymbol{A} : \boldsymbol{B} \\ &\text{tr}\,(\boldsymbol{A}) = \text{tr}\,(\boldsymbol{A} \cdot \boldsymbol{I}) = \boldsymbol{A} : \boldsymbol{I} \end{aligned} \tag{1.1.74}$$

1.1.2.7 Determinant of second order tensors

determinant $\det(\boldsymbol{A})$ of second order tensor $\boldsymbol{A}$ introduces a scalar $\det(\boldsymbol{A}) \in \mathcal{R}$

$$\begin{aligned}\det(\boldsymbol{A}) &= \det(A_{ij}) = \tfrac{1}{6}\, e_{ijk}\, e_{abc}\, A_{ia}\, A_{jb}\, A_{kc} \\ &= A_{11}A_{22}A_{33} + A_{21}A_{32}A_{13} + A_{31}A_{12}A_{23} \\ &- A_{11}A_{23}A_{32} - A_{22}A_{31}A_{13} - A_{33}A_{12}A_{21}\end{aligned} \quad (1.1.75)$$

with

$$III_A = \det(\boldsymbol{A}) \quad (1.1.76)$$

determinant defining vector product $\boldsymbol{u} \times \boldsymbol{v}$

$$\boldsymbol{u} \times \boldsymbol{v} = \det \begin{bmatrix} u_1 & v_1 & \boldsymbol{e}_1 \\ u_2 & v_2 & \boldsymbol{e}_2 \\ u_3 & v_3 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} u_2v_3 & - & u_3v_2 \\ u_3v_1 & - & u_1v_3 \\ u_1v_2 & - & u_2v_1 \end{bmatrix} \quad (1.1.77)$$

determinant defining scalar triple vector product $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$

$$[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}] = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} \quad (1.1.78)$$

properties of determinant of a second order tensors

$$\begin{aligned} &\det(\boldsymbol{I}) = 1 \\ &\det(\boldsymbol{A}^{\mathrm{t}}) = \det(\boldsymbol{A}) \\ &\det(\alpha\, \boldsymbol{A}) = \alpha^3 \det(\boldsymbol{A}) \\ &\det(\boldsymbol{A} \cdot \boldsymbol{B}) = \det(\boldsymbol{A}) \det(\boldsymbol{B}) \\ &\det(\boldsymbol{u} \otimes \boldsymbol{v}) = 0 \end{aligned} \quad (1.1.79)$$

1.1.2.8 Inverse of second order tensors

if $\det(\boldsymbol{A}) \neq 0$
existence of inverse $\boldsymbol{A}^{-1}$ of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \cdot \boldsymbol{A} = \boldsymbol{I} \tag{1.1.80}$$

in particular

$$\boldsymbol{v} = \boldsymbol{A} \cdot \boldsymbol{u} \qquad \boldsymbol{A}^{-1} \cdot \boldsymbol{v} = \boldsymbol{u} \tag{1.1.81}$$

properties of inverse of two second order tensors

$$\begin{aligned} (\boldsymbol{A}^{-1})^{-1} &= \boldsymbol{A} \\ (\alpha\, \boldsymbol{A}^{-1})^{-1} &= \alpha^{-1}\, \boldsymbol{A} \\ (\boldsymbol{A} \cdot \boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1} \cdot \boldsymbol{A}^{-1} \end{aligned} \tag{1.1.82}$$

determinant $\det(\boldsymbol{A}^{-1})$ of inverse of $\boldsymbol{A}$

$$\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A}) \tag{1.1.83}$$

adjoint $\boldsymbol{A}^{\mathrm{adj}}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\mathrm{adj}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-1} \tag{1.1.84}$$

cofactor $\boldsymbol{A}^{\mathrm{cof}}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\mathrm{cof}} = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = (\boldsymbol{A}^{\mathrm{adj}})^{\mathrm{t}} \tag{1.1.85}$$

with

$$\partial_{A} \det(\boldsymbol{A}) = \det(\boldsymbol{A})\, \boldsymbol{A}^{-\mathrm{t}} = III_{A}\, \boldsymbol{A}^{-\mathrm{t}} = \boldsymbol{A}^{\mathrm{cof}} \tag{1.1.86}$$

1.1.3 Spectral decomposition

eigenvalue problem of arbitrary second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} \cdot \boldsymbol{n}_A = \lambda_A \, \boldsymbol{n}_A \qquad [\, \boldsymbol{A} - \lambda_A \, \boldsymbol{I} \,] \cdot \boldsymbol{n}_A = \boldsymbol{0} \tag{1.1.87}$$

solution introduces eigenvector(s) $\boldsymbol{n}_{Ai}$ and eigenvalue(s) λ_{Ai}

$$\det \left(\boldsymbol{A} - \lambda_A \, \boldsymbol{I} \right) = 0 \tag{1.1.88}$$

alternative representation in terms of scalar triple product

$$[\boldsymbol{A} \cdot \boldsymbol{u} - \lambda_A \boldsymbol{u}, \boldsymbol{A} \cdot \boldsymbol{v} - \lambda_A \boldsymbol{v}, \boldsymbol{A} \cdot \boldsymbol{w} - \lambda_A \boldsymbol{w}] = 0 \tag{1.1.89}$$

removal of arbitrary factor $[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}]$ yields characteristic equation

$$\lambda_A^3 - I_A \, \lambda_A^2 + II_A \, \lambda_A - III_A = 0 \tag{1.1.90}$$

roots of characteristic equations are principal invariants of $\boldsymbol{A}$

$$\begin{aligned} I_A &= \operatorname{tr}\left(\boldsymbol{A} \right) \\ II_A &= \tfrac{1}{2} \left[\operatorname{tr}^2 \left(\boldsymbol{A} \right) - \operatorname{tr}\left(\boldsymbol{A}^2 \right) \right] \\ III_A &= \det \left(\boldsymbol{A} \right) \end{aligned} \tag{1.1.91}$$

spectral decomposition of $\boldsymbol{A}$

$$\boldsymbol{A} = \sum_{i=1}^{3} \lambda_{Ai} \, \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai} \tag{1.1.92}$$

Cayleigh–Hamilton theorem:
a tensor $\boldsymbol{A}$ satisfies its own characteristic equation

$$\boldsymbol{A}^3 - I_A \, \boldsymbol{A}^2 + II_A \, \boldsymbol{A} - III_A \, \boldsymbol{I} = \boldsymbol{0} \tag{1.1.93}$$

1.1.4 Symmetric – skew-symmetric decomposition

symmetric – skew-symmetric decomposition of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\rm t}] + \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\rm t}] = \boldsymbol{A}^{\rm sym} + \boldsymbol{A}^{\rm skw} \tag{1.1.94}$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{A}^{\rm sym}$ and $\boldsymbol{A}^{\rm skw}$

$$\boldsymbol{A}^{\rm sym} = (\boldsymbol{A}^{\rm sym})^{\rm t} \qquad \boldsymbol{A}^{\rm skw} = -(\boldsymbol{A}^{\rm skw})^{\rm t} \tag{1.1.95}$$

• symmetric second order tensor $\boldsymbol{A}^{\rm sym}$

$$\boldsymbol{A}^{\rm sym} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\rm t}] = \mathbb{I}^{\rm sym} : \boldsymbol{A} \tag{1.1.96}$$

upon double contraction symmetric fourth order unit tensor $\mathbb{I}^{\rm sym}$ extracts symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\rm sym} &= \tfrac{1}{2}\,[\mathbb{I} + \mathbb{I}^{\rm t}] \\ \mathbb{I}^{\rm sym} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.97}$$

• skew–symmetric second order tensor $\boldsymbol{A}^{\rm skw}$

$$\boldsymbol{A}^{\rm skw} = \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\rm t}] = \mathbb{I}^{\rm skw} : \boldsymbol{A} \tag{1.1.98}$$

upon double contraction skew-symmetric fourth order unit tensor $\mathbb{I}^{\rm skw}$ extracts skew-symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\rm skw} &= \tfrac{1}{2}\,[\mathbb{I} - \mathbb{I}^{\rm t}] \\ \mathbb{I}^{\rm skw} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.99}$$

1.1.4.1 Symmetric tensors

symmetric part $\boldsymbol{A}^{\rm sym}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\rm sym} = \frac{1}{2}[\boldsymbol{A} + \boldsymbol{A}^{\rm t}] \qquad \boldsymbol{A}^{\rm sym} = (\boldsymbol{A}^{\rm sym})^{\rm t} \tag{1.1.100}$$

alternative representation

$$\boldsymbol{A}^{\rm sym} = \mathbb{I}^{\rm sym} : \boldsymbol{A} \tag{1.1.101}$$

whereby symmetric fourth order tensor $\mathbb{I}^{\rm sym}$ extracts symmetric part $\boldsymbol{A}^{\rm sym}$ of second order tensor $\boldsymbol{A}$

a symmetric second order tensor $\boldsymbol{S} = \boldsymbol{A}^{\rm sym}$ processes three real eigenvalues $\{\lambda_{Si}\}_{i=1,2,3}$ and three corresponding orthogonal eigenvectors $\{\boldsymbol{n}_{Si}\}_{i=1,2,3}$, such that the spectral representation of $\boldsymbol{S}$ takes the following form

$$\boldsymbol{S} = \sum_{i=1}^{3} \lambda_{Si} \, (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \tag{1.1.102}$$

three invariants I_S, II_S, III_S of symmetric tensor $\boldsymbol{S} = \boldsymbol{A}^{\rm sym}$

$$\begin{aligned} I_S &= \lambda_{S1} + \lambda_{S2} + \lambda_{S3} \\ II_S &= \lambda_{S2}\,\lambda_{S3} + \lambda_{S3}\,\lambda_{S1} + \lambda_{S1}\,\lambda_{S2} \\ III_S &= \lambda_{S1}\,\lambda_{S2}\,\lambda_{S3} \end{aligned} \tag{1.1.103}$$

square root $\sqrt{\boldsymbol{S}}$, inverse $\boldsymbol{S}^{-1}$, exponent $\exp(\boldsymbol{S})$ and logarithm $\ln(\boldsymbol{S})$, of positive semi-definite symmetric tensor $\boldsymbol{S}$ for which $\lambda_{Si} \geq 0$

$$\begin{aligned} \sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3} \sqrt{\lambda_{Si}} \; (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \boldsymbol{S}^{-1} &= \textstyle\sum_{i=1}^{3} \lambda_{Si}^{-1} \; (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \exp(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \exp(\lambda_{Si}) \; (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \\ \ln(\boldsymbol{S}) &= \textstyle\sum_{i=1}^{3} \ln(\lambda_{Si}) \; (\boldsymbol{n}_{Si} \otimes \boldsymbol{n}_{Si}) \end{aligned} \tag{1.1.104}$$

1.1.4.2 Skew-symmetric tensors

skew-symmetric part $\boldsymbol{A}^{\rm skw}$ of a second order tensor $\boldsymbol{A}$

$$\boldsymbol{A}^{\rm skw} = \frac{1}{2}[\boldsymbol{A} - \boldsymbol{A}^{\rm t}] \qquad \boldsymbol{A}^{\rm skw} = -(\boldsymbol{A}^{\rm skw})^{\rm t} \tag{1.1.105}$$

alternative representation

$$\boldsymbol{A}^{\rm skw} = \mathbb{I}^{\rm skw} : \boldsymbol{A} \tag{1.1.106}$$

whereby skew-symmetric fourth order tensor $\mathbb{I}^{\rm skw}$ extracts skew-symmetric part $\boldsymbol{A}^{\rm skw}$ of second order tensor $\boldsymbol{A}$

a skew-symmetric second order tensor $\boldsymbol{W} = \boldsymbol{A}^{\rm skw}$ posses three independent entries, three entries vanish identically, three are equal to the negative of the independent entries, these define the axial vector $\boldsymbol{w}$

$$\boldsymbol{w} = -\frac{1}{2}\overset{3}{\boldsymbol{e}} : \boldsymbol{W} \qquad \boldsymbol{w} = -\overset{3}{\boldsymbol{e}} \cdot \boldsymbol{w} \tag{1.1.107}$$

associated with each skew-symmetric tensor $\boldsymbol{W} = \boldsymbol{A}^{\rm skw}$

$$\boldsymbol{W} \cdot \boldsymbol{p} = \boldsymbol{w} \times \boldsymbol{p} \tag{1.1.108}$$

three invariants I_W, II_W, III_W of skew-symmetric tensor $\boldsymbol{W}$

$$\begin{aligned} I_W &= \mathrm{tr}(\boldsymbol{W}) &&= 0 \\ II_W &= \boldsymbol{w} \cdot \boldsymbol{w} && \\ III_W &= \det(\boldsymbol{W}) &&= 0 \end{aligned} \tag{1.1.109}$$

1.1.5 Volumetric – deviatoric decomposition

volumetric – deviatoric decomposition of second order tensor $\boldsymbol{A}$

$$\boldsymbol{A} = \boldsymbol{A}^{\text{vol}} + \boldsymbol{A}^{\text{dev}} \tag{1.1.110}$$

with volumetric and deviatoric second order tensor $\boldsymbol{A}^{\text{vol}}$ and $\boldsymbol{A}^{\text{dev}}$

$$\text{tr}(\boldsymbol{A}^{\text{vol}}) = \text{tr}(\boldsymbol{A}) \qquad \text{tr}(\boldsymbol{A}^{\text{dev}}) = 0 \tag{1.1.111}$$

• volumetric second order tensor $\boldsymbol{A}^{\text{vol}}$

$$\boldsymbol{A}^{\text{vol}} = \frac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\, \boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{A} \tag{1.1.112}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\text{vol}}$ extracts volumetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\, \boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\, \delta_{ij}\delta_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.113}$$

• deviatoric second order tensor $\boldsymbol{A}^{\text{dev}}$

$$\boldsymbol{A}^{\text{dev}} = \boldsymbol{A} - \frac{1}{3}[\boldsymbol{A} : \boldsymbol{I}]\, \boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{A} = 0 \tag{1.1.114}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\text{dev}}$ extracts deviatoric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \left[\tfrac{1}{2}\, \delta_{ik}\delta_{jl} + \tfrac{1}{2}\, \delta_{il}\delta_{jk} - \tfrac{1}{3}\, \delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{1.1.115}$$

1.1.6 Orthogonal tensors

a second order tensor $\boldsymbol{Q}$ is called orthogonal if its inverse $\boldsymbol{Q}^{-1}$ is identical to its transpose $\boldsymbol{Q}^{\mathrm{t}}$

$$\boldsymbol{Q}^{-1} = \boldsymbol{Q}^{\mathrm{t}} \qquad \Leftrightarrow \qquad \boldsymbol{Q}^{\mathrm{t}} \cdot \boldsymbol{Q} = \boldsymbol{Q} \cdot \boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{I} \tag{1.1.116}$$

a second order tensor $\boldsymbol{A}$ can be decomposed multiplicatively into a positive definite symmetric tensor $\boldsymbol{U}^{\mathrm{t}} = \boldsymbol{U}$ or $\boldsymbol{V}^{\mathrm{t}} = \boldsymbol{V}$ with $\boldsymbol{a} \cdot \boldsymbol{U} \cdot \boldsymbol{a} \geq 0$ and $\boldsymbol{a} \cdot \boldsymbol{V} \cdot \boldsymbol{a} \geq 0$ and an orthogonal tensor $\boldsymbol{Q}^{\mathrm{t}} = \boldsymbol{Q}^{-1}$ as

$$\boldsymbol{A} = \boldsymbol{Q} \cdot \boldsymbol{U} = \boldsymbol{V} \cdot \boldsymbol{Q} \tag{1.1.117}$$

with S0(3) being the special orthogonal group, $\boldsymbol{Q} \in$ S0(3) if $\det(\boldsymbol{Q}) = +1$, then $\boldsymbol{Q}$ is called proper orthogonal

a proper orthogonal tensor $\boldsymbol{Q} \in$ S0(3) has an eigenvalue equal to one $\lambda_Q = 1$ introducing an eigenvector $\boldsymbol{n}_Q$ such that

$$\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q \tag{1.1.118}$$

let $\{\boldsymbol{n}_{Qi}\}_{i=1,2,3}$ be a Cartesian basis containing the vector $\boldsymbol{n}_Q$, then matrix representation of coordinates Q_{ij}

$$[Q_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +\cos\varphi & +\sin\varphi \\ 0 & -\sin\varphi & +\cos\varphi \end{bmatrix} \tag{1.1.119}$$

geometric interpretation: $\boldsymbol{Q}$ characterizes a finite rotation around the axis $\boldsymbol{n}_Q$ with $\boldsymbol{Q} \cdot \boldsymbol{n}_Q = \boldsymbol{n}_Q$, i.e. associated with $\lambda_Q = 1$

1.2 Tensor analysis

1.2.1 Derivatives

consider smooth, differentiable scalar field Φ with

- scalar argument $\Phi : \mathcal{R} \to \mathcal{R}; \quad \Phi(x) = \alpha$
- vectorial argument $\Phi : \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(\boldsymbol{x}) = \alpha$
- tensorial argument $\Phi : \mathcal{R}^3 \times \mathcal{R}^3 \to \mathcal{R}; \quad \Phi(\boldsymbol{X}) = \alpha$

1.2.1.1 Frechet derivative

$$
\begin{aligned}
&\bullet\ \text{scalar} && \mathrm{D}\,\Phi(x) = \frac{\partial \Phi(x)}{\partial x} = \partial_x \Phi(x)\\
&\bullet\ \text{vectorial} && \mathrm{D}\,\Phi(\boldsymbol{x}) = \frac{\partial \Phi(\boldsymbol{x})}{\partial \boldsymbol{x}} = \partial_{\boldsymbol{x}} \Phi(\boldsymbol{x})\\
&\bullet\ \text{tensorial} && \mathrm{D}\,\Phi(\boldsymbol{X}) = \frac{\partial \Phi(\boldsymbol{X})}{\partial \boldsymbol{X}} = \partial_{\boldsymbol{X}} \Phi(\boldsymbol{X})
\end{aligned}
\tag{1.2.1}
$$

1.2.1.2 Gateaux derivative

Gateaux derivative as particular Frechet derivative with respect to directions u, $\boldsymbol{u}$ and $\boldsymbol{U}$

$$
\begin{aligned}
&\bullet\ \text{scalar} && \mathrm{D}\,\Phi(x)\ \ u = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(x + \epsilon\, u)\,|_{\epsilon=0} \quad \forall\, u \in \mathcal{R}\\
&\bullet\ \text{vectorial} && \mathrm{D}\,\Phi(\boldsymbol{x}) \cdot \boldsymbol{u} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{x} + \epsilon\, \boldsymbol{u})\,|_{\epsilon=0} \quad \forall\, \boldsymbol{u} \in \mathcal{R}^3\\
&\bullet\ \text{tensorial} && \mathrm{D}\,\Phi(\boldsymbol{X}) : \boldsymbol{U} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{X} + \epsilon\, \boldsymbol{U})|_{\epsilon=0} \quad \forall\, \boldsymbol{U} \in \mathcal{R}^3 \otimes \mathcal{R}^3
\end{aligned}
\tag{1.2.2}
$$

in what follows in particular vectorial arguments, i.e. point position $\boldsymbol{x}$ or displacement $\boldsymbol{u}$

1.2.2 Gradient

consider scalar– and vector–valued field $f(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \rightarrow \mathcal{R} \qquad f : \boldsymbol{x} \rightarrow f(\boldsymbol{x})$$

$$\boldsymbol{f} : \mathcal{B} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \rightarrow \boldsymbol{f}(\boldsymbol{x})$$

1.2.2.1 Gradient of a scalar–valued function

gradient $\nabla f(\boldsymbol{x})$ of scalar–valued field $f(\boldsymbol{x})$

$$\nabla f(\boldsymbol{x}) = \frac{\partial f(\boldsymbol{x})}{\partial x_i} = f_{,i}(\boldsymbol{x})\, \boldsymbol{e}_i \tag{1.2.3}$$

and thus

$$\nabla f(\boldsymbol{x}) = \begin{bmatrix} f_{,1} \\ f_{,2} \\ f_{,3} \end{bmatrix} \tag{1.2.4}$$

gradient of scalar–valued field renders a vector field

1.2.2.2 Gradient of a vector–valued function

gradient $\nabla \boldsymbol{f}(\boldsymbol{x})$ of vector–valued field $\boldsymbol{f}(\boldsymbol{x})$

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \frac{\partial f_i(\boldsymbol{x})}{\partial x_j} = f_{i,j}(\boldsymbol{x})\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{1.2.5}$$

and thus

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} \\ f_{2,1} & f_{2,2} & f_{2,3} \\ f_{3,1} & f_{3,2} & f_{3,3} \end{bmatrix} \tag{1.2.6}$$

gradient of vector–valued field renders a (second order) tensor field

1.2.3 Divergence

consider vector and second order tensor field $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{F}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}\ (\boldsymbol{x})$$
$$\boldsymbol{F} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \qquad \boldsymbol{F} : \boldsymbol{x} \to \boldsymbol{F}\ (\boldsymbol{x})$$

1.2.3.1 Divergence of a vector field

divergence $\nabla \boldsymbol{f}\ (\boldsymbol{x})$ of vector field $\boldsymbol{f}\ (\boldsymbol{x})$

$$\mathrm{div}(\boldsymbol{f}\ (\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{f}\ (\boldsymbol{x})) = \nabla \boldsymbol{f}\ (\boldsymbol{x}) : \mathbf{1} \tag{1.2.7}$$

with $\nabla \boldsymbol{f}\ (\boldsymbol{x}) = f_{i,j}(\boldsymbol{x})\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j$

$$\mathrm{div}(\boldsymbol{f}\ (\boldsymbol{x})) = f_{i,i}(\boldsymbol{x}) = f_{1,1} + f_{2,2} + f_{3,3} \tag{1.2.8}$$

divergence of a vector field renders a scalar field

1.2.3.2 Divergence of a tensor field

divergence $\nabla \boldsymbol{F}\ (\boldsymbol{x})$ of tensor field $\boldsymbol{F}\ (\boldsymbol{x})$

$$\mathrm{div}(\boldsymbol{F}\ (\boldsymbol{x})) = \mathrm{tr}(\nabla \boldsymbol{F}\ (\boldsymbol{x})) = \nabla \boldsymbol{F}\ (\boldsymbol{x}) : \mathbf{1} \tag{1.2.9}$$

with $\nabla \boldsymbol{F}\ (\boldsymbol{x}) = F_{i,jk}(\boldsymbol{x})\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k$

$$\mathrm{div}(\boldsymbol{F}\ (\boldsymbol{x})) = F_{ij,j}(\boldsymbol{x}) = \begin{bmatrix} F_{11,1} + F_{12,2} + F_{13,3} \\ F_{21,1} + F_{22,2} + F_{23,3} \\ F_{31,1} + F_{32,2} + F_{33,3} \end{bmatrix} \tag{1.2.10}$$

divergence of a second order tensor field renders a vector field

1.2.4 Laplace operator

consider scalar– and vector–valued field $f(\boldsymbol{x})$ and $\boldsymbol{f}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$f : \mathcal{B} \to \mathcal{R} \qquad f : \boldsymbol{x} \to f(\boldsymbol{x})$$
$$\boldsymbol{f} : \mathcal{B} \to \mathcal{R}^3 \qquad \boldsymbol{f} : \boldsymbol{x} \to \boldsymbol{f}(\boldsymbol{x})$$

1.2.4.1 Laplace operator acting on scalar–valued function

Laplace operator $\Delta f(\boldsymbol{x})$ acting on scalar–valued field $f(\boldsymbol{x})$

$$\Delta f(\boldsymbol{x}) = \operatorname{div}(\nabla(f(\boldsymbol{x}))) \tag{1.2.11}$$

and thus

$$\Delta f(\boldsymbol{x}) = f_{,ii} = f_{,11} + f_{,22} + f_{,33} \tag{1.2.12}$$

Laplace operator acting on on scalar–valued field renders a scalar field

1.2.4.2 Laplace operator acting on vector–valued function

Laplace operator $\Delta \boldsymbol{f}(\boldsymbol{x})$ acting on vector–valued field $\boldsymbol{f}(\boldsymbol{x})$

$$\Delta \boldsymbol{f}(\boldsymbol{x}) = \operatorname{div}(\nabla(\boldsymbol{f}(\boldsymbol{x}))) \tag{1.2.13}$$

and thus

$$\Delta \boldsymbol{f}(\boldsymbol{x}) = f_{i,jj} = \begin{bmatrix} f_{1,11} + f_{1,22} + f_{1,33} \\ f_{2,11} + f_{2,22} + f_{2,33} \\ f_{3,11} + f_{3,22} + f_{3,33} \end{bmatrix} \tag{1.2.14}$$

Laplace operator acting on on vector–valued field renders a vector field

Useful transformation formulae

consider scalar, vector and second order tensor field $\alpha(\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{A}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$
\begin{array}{ll}
\alpha : \mathcal{B} \to \mathcal{R} & \alpha : \boldsymbol{x} \to \alpha\;(\boldsymbol{x}) \\
\boldsymbol{u} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{u} : \boldsymbol{x} \to \boldsymbol{u}\;(\boldsymbol{x}) \\
\boldsymbol{v} : \mathcal{B} \to \mathcal{R}^3 & \boldsymbol{v} : \boldsymbol{x} \to \boldsymbol{v}\;(\boldsymbol{x}) \\
\boldsymbol{A} : \mathcal{B} \to \mathcal{R}^3 \otimes \mathcal{R}^3 & \boldsymbol{A} : \boldsymbol{x} \to \boldsymbol{A}\;(\boldsymbol{x})
\end{array}
$$

important transformation formulae

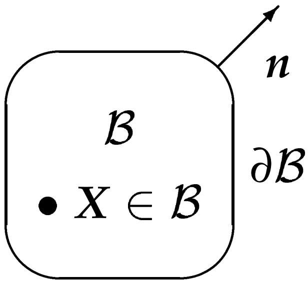
$$
\begin{aligned}
\nabla\;(\alpha\,\boldsymbol{u}) &= \boldsymbol{u} \otimes \nabla\alpha + \alpha\,\nabla\boldsymbol{u} \\
\nabla\;(\boldsymbol{u} \cdot \boldsymbol{v}) &= \boldsymbol{u} \cdot \nabla\boldsymbol{v} + \boldsymbol{v} \cdot \nabla\boldsymbol{u} \\
\mathrm{div}\;(\alpha\,\boldsymbol{u}) &= \alpha\,\mathrm{div}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla\alpha \\
\mathrm{div}\;(\alpha\,\boldsymbol{A}) &= \alpha\,\mathrm{div}(\boldsymbol{A}) + \boldsymbol{A} \cdot \nabla\alpha \\
\mathrm{div}\;(\boldsymbol{u} \cdot \boldsymbol{A}) &= \boldsymbol{u} \cdot \mathrm{div}(\boldsymbol{A}) + \boldsymbol{A} : \nabla\boldsymbol{u} \\
\mathrm{div}\;(\boldsymbol{u} \otimes \boldsymbol{v}) &= \boldsymbol{u}\,\mathrm{div}(\boldsymbol{v}) + \boldsymbol{v} \cdot \nabla\boldsymbol{u}^{\mathrm{t}}
\end{aligned}
\tag{1.2.15}
$$

or in index notation

$$
\begin{aligned}
(\alpha\,u_i)_{,j} &= u_i\,\alpha_{,j} + \alpha\,u_{i,j} \\
(u_i\,v_i)_{,j} &= u_i\,v_{i,j} + v_i\,u_{i,j} \\
(\alpha\,u_i)_{,i} &= \alpha\,u_{i,i} + u_i\,\alpha_{,i} \\
(\alpha\,A_{ij})_{,j} &= \alpha\,A_{ij,j} + A_{ij}\,\alpha_{,j} \\
(u_i\,A_{ij})_{,j} &= u_i\,A_{ij,j} + A_{ij}\,u_{i,j} \\
(u_i\,v_j)_{,j} &= u_i\,v_{j,j} + v_j\,u_{i,j}
\end{aligned}
\tag{1.2.16}
$$

1.2.5 Integral transformations

integral theorems define relations
between surface integral $\int_{\partial\mathcal{B}} ...\mathrm{d}A$
and volume integral $\int_{\mathcal{B}} ...\mathrm{d}V$



consider scalar, vector and second order tensor field $\alpha(\boldsymbol{x})$, $\boldsymbol{u}(\boldsymbol{x})$ and $\boldsymbol{A}(\boldsymbol{x})$ on domain $\mathcal{B} \in \mathcal{R}^3$

$$\alpha : \mathcal{B} \rightarrow \mathcal{R} \qquad \alpha : \boldsymbol{x} \rightarrow \alpha \ (\boldsymbol{x})$$
$$\boldsymbol{u} : \mathcal{B} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{u} : \boldsymbol{x} \rightarrow \boldsymbol{u} \ (\boldsymbol{x})$$
$$\boldsymbol{A} : \mathcal{B} \rightarrow \mathcal{R}^3 \otimes \mathcal{R}^3 \qquad \boldsymbol{A} : \boldsymbol{x} \rightarrow \boldsymbol{A} \ (\boldsymbol{x})$$

1.2.5.1 Integral theorem for scalar–valued fields (Green theorem)

$$\begin{aligned} \int_{\partial\mathcal{B}} \alpha\, \boldsymbol{n} \ \mathrm{d}A &= \int_{\mathcal{B}} \nabla \alpha \ \ \mathrm{d}V \\ \int_{\partial\mathcal{B}} \alpha\, n_i \ \mathrm{d}A &= \int_{\mathcal{B}} \ \ \alpha_{,i} \ \mathrm{d}V \end{aligned} \tag{1.2.17}$$

1.2.5.2 Integral theorem for vector–valued fields (Gauss theorem)

$$\begin{aligned} \int_{\partial\mathcal{B}} \boldsymbol{u} \cdot \boldsymbol{n} \ \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{u}) \ \mathrm{d}V \\ \int_{\partial\mathcal{B}} u_i\, n_i \ \mathrm{d}A &= \int_{\mathcal{B}} \ \ u_{i,i} \ \ \mathrm{d}V \end{aligned} \tag{1.2.18}$$

1.2.5.3 Integral theorem for tensor–valued fields (Gauss theorem)

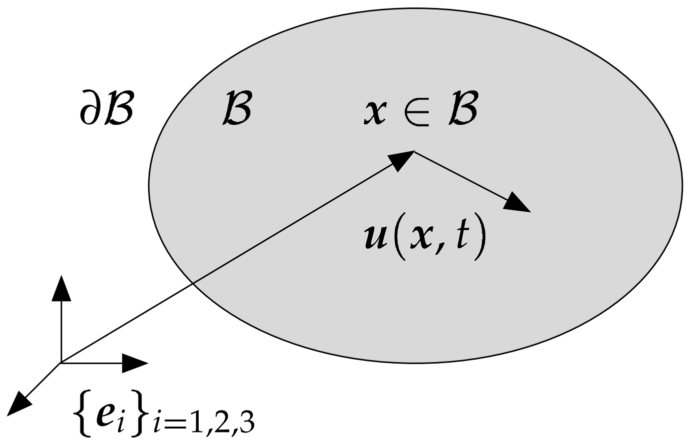
$$\begin{aligned} \int_{\partial\mathcal{B}} \boldsymbol{A} \cdot \boldsymbol{n} \ \ \mathrm{d}A &= \int_{\mathcal{B}} \mathrm{div}(\boldsymbol{A}) \ \mathrm{d}V \\ \int_{\partial\mathcal{B}} A_{ij}\, n_j \ \mathrm{d}A &= \int_{\mathcal{B}} \ \ A_{ij,j} \ \ \mathrm{d}V \end{aligned} \tag{1.2.19}$$

2 Kinematics

restriction to geometrically linear theory, valid if local strains remain small

2.1 Motion

consider a material body $\mathcal{B}$ as a simply connected subset of the Euclidian space $\mathcal{R}^3$ as $\mathcal{B} \subset \mathcal{R}^3$, with the boundary being denoted as $\partial\mathcal{B}$, a material point is defined as a point of the body $\boldsymbol{x} \in \mathcal{B}$



motion of a body $\mathcal{B} \subset \mathcal{R}^3$ characterized through time dependent vector field of displacements $\boldsymbol{u} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{u} : \mathcal{B} \times \mathcal{R} \to \mathcal{R}^3 \qquad \boldsymbol{u}(\boldsymbol{x}, t) = u_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.1.1}$$

2.2 Rates of kinematic quantities

2.2.1 Velocity

vector field of velocities $\boldsymbol{v} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{v} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{v}(\boldsymbol{x}, t) = v_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.2.1}$$

velocity field $\boldsymbol{v}$ defined through rate of change of displacement field $\boldsymbol{u}$

$$\boldsymbol{v}(\boldsymbol{x}, t) = \mathrm{D}_t \boldsymbol{u}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t}\right|_{x \text{fixed}} \tag{2.2.2}$$

velocity vector

$$\boldsymbol{v} = [v_1, v_2, v_3]^{\mathrm{t}} = \mathrm{D}_t [u_1, u_2, u_3]^{\mathrm{t}} \tag{2.2.3}$$

common notation in the literature $\boldsymbol{v} = \dot{\boldsymbol{u}}$

2.2.2 Acceleration

vector field of accelerations $\boldsymbol{a} \in \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{a} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \qquad \boldsymbol{a}(\boldsymbol{x}, t) = a_i(\boldsymbol{x}, t)\, \boldsymbol{e}_i \tag{2.2.4}$$

acceleration field $\boldsymbol{a}$ defined through rate of change of velocity field $\boldsymbol{v}$

$$\boldsymbol{a}(\boldsymbol{x}, t) = \mathrm{D}_t \boldsymbol{v}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{v}(\boldsymbol{x}, t)}{\partial t}\right|_{x \text{fixed}} = \left.\frac{\partial^2 \boldsymbol{u}(\boldsymbol{x}, t)}{\partial t^2}\right|_{x \text{fixed}} \tag{2.2.5}$$

acceleration vector

$$\boldsymbol{a} = [a_1, a_2, a_3]^{\mathrm{t}} = \mathrm{D}_t [v_1, v_2, v_3]^{\mathrm{t}} = \mathrm{D}_t^2 [u_1, u_2, u_3]^{\mathrm{t}} \tag{2.2.6}$$

common notation in the literature $\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{\boldsymbol{u}}$

2.3 Gradients of kinematic quantities

2.3.1 Displacement gradient

second order tensor field of displacement gradient $\boldsymbol{H} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{H} : \mathcal{B} \times \mathcal{R} \rightarrow \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{H}(\boldsymbol{x}, t) = H_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.1)$$

displacement gradient $\boldsymbol{H}$ defined through gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{H}(\boldsymbol{x}, t) = \nabla \boldsymbol{u}(\boldsymbol{x}, t) = \left.\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}\right|_{t\text{fixed}} \quad (2.3.2)$$

index representation

$$H_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{\partial u_i}{\partial X_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = u_{i,j}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.3)$$

matrix representation of coordinates H_{ij}

$$[H_{ij}] = \begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & u_{3,3} \end{bmatrix} \quad (2.3.4)$$

• displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$ is non–symmetric, $\boldsymbol{H} \neq \boldsymbol{H}^{\mathrm{t}}$

• displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$ does not vanish for pointwise rigid body motion

Symmetric-skew-symmetric decomposition of displacement gradient

symmetric–skew-symmetric decomposition of displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$

$$\boldsymbol{H} = \frac{1}{2}[\boldsymbol{H} + \boldsymbol{H}^{\text{t}}] + \frac{1}{2}[\boldsymbol{H} - \boldsymbol{H}^{\text{t}}] = \boldsymbol{\epsilon} + \boldsymbol{\omega} \tag{2.3.5}$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{\epsilon} = \boldsymbol{H}^{\text{sym}}$ and $\boldsymbol{\omega} = \boldsymbol{H}^{\text{skw}}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\text{t}} \qquad \boldsymbol{\omega} = -\boldsymbol{\omega}^{\text{t}} \tag{2.3.6}$$

• geometrically linear strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \frac{1}{2}[\boldsymbol{H} + \boldsymbol{H}^{\text{t}}] = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^{\text{t}} \boldsymbol{u}] = \nabla^{\text{sym}} \boldsymbol{u} = \mathbb{I}^{\text{sym}} : \nabla \boldsymbol{u} \tag{2.3.7}$$

upon double contraction symmetric fourth order unit tensor $\mathbb{I}^{\text{sym}}$ extracts symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{sym}} &= \tfrac{1}{2}\,[\mathbb{I} + \mathbb{I}^{\text{t}}] \\ \mathbb{I}^{\text{sym}} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.8}$$

• geometrically linear rotation tensor $\boldsymbol{\omega}$

$$\boldsymbol{\omega} = \frac{1}{2}[\boldsymbol{H} - \boldsymbol{H}^{\text{t}}] = \frac{1}{2}[\nabla \boldsymbol{u} - \nabla^{\text{t}} \boldsymbol{u}] = \nabla^{\text{skw}} \boldsymbol{u} = \mathbb{I}^{\text{skw}} : \nabla \boldsymbol{u} \tag{2.3.9}$$

upon double contraction skew-symmetric fourth order unit tensor $\mathbb{I}^{\text{skw}}$ extracts skew-symmetric part of second order tensor

$$\begin{aligned} \mathbb{I}^{\text{skw}} &= \tfrac{1}{2}\,[\mathbb{I} - \mathbb{I}^{\text{t}}] \\ \mathbb{I}^{\text{skw}} &= \tfrac{1}{2}\,[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.10}$$

2.3.2 Strain

second order tensor field of (geometrically linear) strains $\boldsymbol{\epsilon} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{\epsilon} : \mathcal{B} \times \mathcal{R} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{\epsilon}(\boldsymbol{x}, t) = \epsilon_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.11)$$

strain tensor $\boldsymbol{\epsilon}$ defined through symmetric part of gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{\epsilon}(\boldsymbol{x}, t) = \nabla^{\text{sym}} \boldsymbol{u}(\boldsymbol{x}, t) = \left[\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \right]^{\text{sym}} \quad (2.3.12)$$

index representation

$$\epsilon_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{1}{2} [u_{i,j} + u_{j,i}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.13)$$

matrix representation of coordinates ϵ_{ij}

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2u_{1,1} & u_{1,2} + u_{2,1} & u_{1,3} + u_{3,1} \\ u_{2,1} + u_{1,2} & 2u_{2,2} & u_{2,3} + u_{3,2} \\ u_{3,1} + u_{1,3} & u_{3,2} + u_{2,3} & 2u_{3,3} \end{bmatrix} \quad (2.3.14)$$

strain tensor $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ is symmetric, $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\text{t}}$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{21} & \epsilon_{31} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{32} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} = \epsilon_{ji} \quad (2.3.15)$$

ϵ_{ij} for $i = j$ ... diagonal entries: normal strain
ϵ_{ij} for $i \neq j$ ... off–diagonal entries: shear strain

2.3.3 Rotation

second order tensor field of (geometrically linear) rotation $\boldsymbol{\omega} \in \mathcal{R}^3 \otimes \mathcal{R}^3$ parameterized in terms of position $\boldsymbol{x} \in \mathcal{B}$ and time $t \in \mathcal{R}$

$$\boldsymbol{\omega} : \mathcal{B} \times \mathcal{R} \to \mathcal{R}^3 \otimes \mathcal{R}^3 \quad \boldsymbol{\omega}(\boldsymbol{x}, t) = \omega_{ij}(\boldsymbol{x}, t)\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.16)$$

rotation tensor $\boldsymbol{\omega}$ defined through skew–symmetric part of gradient of displacement field $\boldsymbol{u}$

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \nabla^{\text{skw}} \boldsymbol{u}(\boldsymbol{x}, t) = \left[\frac{\partial \boldsymbol{u}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}} \right]^{\text{skw}} \quad (2.3.17)$$

index representation

$$\omega_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{1}{2} [u_{i,j} - u_{j,i}] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \quad (2.3.18)$$

matrix representation of coordinates ω_{ij}

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & u_{1,2} - u_{2,1} & u_{1,3} - u_{3,1} \\ u_{2,1} - u_{1,2} & 0 & u_{2,3} - u_{3,2} \\ u_{3,1} - u_{1,3} & u_{3,2} - u_{2,3} & 0 \end{bmatrix} \quad (2.3.19)$$

rotation tensor $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$ is skew-symmetric, $\boldsymbol{\omega} = -\boldsymbol{\omega}^{\text{t}}$

$$[\omega_{ij}] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} = - \begin{bmatrix} 0 & \omega_{21} & \omega_{31} \\ \omega_{12} & 0 & \omega_{32} \\ \omega_{13} & \omega_{23} & 0 \end{bmatrix} = -\omega_{ji} \quad (2.3.20)$$

corresponding axial vector $\boldsymbol{w} = -1/2 \overset{3}{\boldsymbol{e}} : \boldsymbol{\omega}$

$$\boldsymbol{w} = [w_1, w_2, w_3]^{\text{t}} = -[\omega_{23}, \omega_{31}, \omega_{12}]^{\text{t}} \quad (2.3.21)$$

Symmetric-skew-symmetric decomposition of displacement gradient

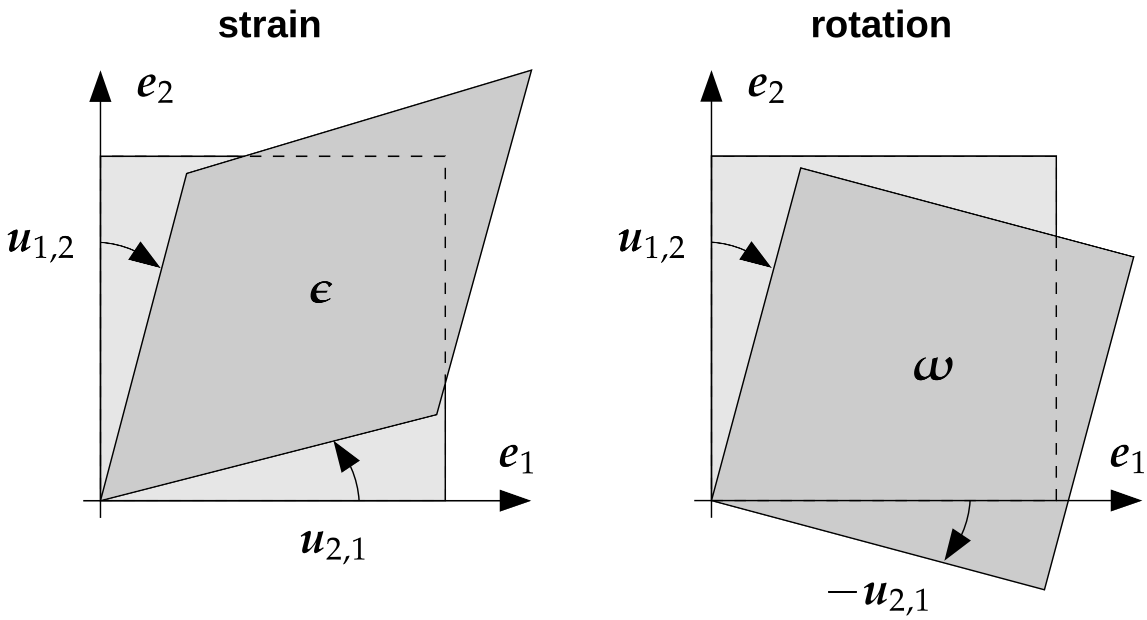
symmetric–skew-symmetric decomposition of displacement gradient $\boldsymbol{H} = \nabla \boldsymbol{u}$

$$\boldsymbol{H} = \frac{1}{2}[\nabla \boldsymbol{u} + \nabla^{t} \boldsymbol{u}] + \frac{1}{2}[\nabla \boldsymbol{u} - \nabla^{t} \boldsymbol{u}] = \boldsymbol{\epsilon} + \boldsymbol{\omega} \qquad (2.3.22)$$

with symmetric and skew-symmetric second order tensor $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ and $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{t} \qquad \boldsymbol{\omega} = -\boldsymbol{\omega}^{t} \qquad (2.3.23)$$

geometric interpretation:
representation of symmetric and skew–symmetric part of displacement gradient for two-dimensional case



$\epsilon_{12} = \frac{1}{2}\,[u_{1,2} + u_{2,1}]$ $\qquad$ $\omega_{12} = \frac{1}{2}\,[u_{1,2} - u_{2,1}]$

symmetric part $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ represents strain while skew-symmetric $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$ part represents rotation

2.3.4 Volumetric–deviatoric decomposition of strain tensor

a material volume element can deform volumetrically and deviatorically, volumetric deformation conserves the shape (i.e. no changes in angles, no sliding) while deviatoric (isochoric) deformation conserves the volume

volumetric – deviatoric decomposition of strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\rm vol} + \boldsymbol{\epsilon}^{\rm dev} \tag{2.3.24}$$

with volumetric and deviatoric strain tensor $\boldsymbol{\epsilon}^{\rm vol}$ and $\boldsymbol{\epsilon}^{\rm dev}$

$$\operatorname{tr}(\boldsymbol{\epsilon}^{\rm vol}) = \operatorname{tr}(\boldsymbol{\epsilon}) \qquad \operatorname{tr}(\boldsymbol{\epsilon}^{\rm dev}) = 0 \tag{2.3.25}$$

• volumetric second order tensor $\boldsymbol{\epsilon}^{\rm vol}$

$$\boldsymbol{\epsilon}^{\rm vol} = \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\rm vol} : \boldsymbol{\epsilon} \tag{2.3.26}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\rm vol}$ extracts volumetric part $\boldsymbol{\epsilon}^{\rm vol}$ of strain tensor

$$\begin{aligned} \mathbb{I}^{\rm vol} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\rm vol} &= \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.27}$$

• deviatoric second order tensor $\boldsymbol{\epsilon}^{\rm dev}$

$$\boldsymbol{\epsilon}^{\rm dev} = \boldsymbol{\epsilon} - \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\rm dev} : \boldsymbol{\epsilon} \tag{2.3.28}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\rm dev}$ extracts deviatoric part of strain tensor

$$\begin{aligned} \mathbb{I}^{\rm dev} &= \mathbb{I}^{\rm sym} - \mathbb{I}^{\rm vol} = \mathbb{I}^{\rm sym} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\rm dev} &= \left[\tfrac{1}{2}\,\delta_{ik}\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\delta_{jk} - \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{2.3.29}$$

Volumetric deformation

volumetric deformation is characterized through the volume dilatation $e \in \mathcal{R}$, i.e. difference of deformed volume and original volume $\mathrm{d}v - \mathrm{d}V$ scaled by original volume $\mathrm{d}V$

$$e = \frac{\mathrm{d}v - \mathrm{d}V}{\mathrm{d}V} \begin{array}{l} = (1+\epsilon_{11})(1+\epsilon_{22})(1+\epsilon_{33}) - 1 \\ = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \mathcal{O}(\epsilon_{ij}^2) \end{array} \qquad (2.3.30)$$

neglection of higher order terms: trace of strain tensor $\mathrm{tr}\,(\boldsymbol{\epsilon}) = \boldsymbol{\epsilon} : \boldsymbol{I} \in \mathcal{R}$ as characteristic measure for volume changes

$$e = \mathrm{div}\, \boldsymbol{u} = \nabla \boldsymbol{u} : \boldsymbol{I} = \boldsymbol{\epsilon} : \boldsymbol{I} = \mathrm{tr}\,(\boldsymbol{\epsilon}) \qquad (2.3.31)$$

volumetric part $\boldsymbol{\epsilon}^{\mathrm{vol}}$ of strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \tfrac{1}{3}\, e\, \boldsymbol{I} = \tfrac{1}{3}\, [\boldsymbol{\epsilon} : \boldsymbol{I}]\, \boldsymbol{I} = \tfrac{1}{3}\, [\boldsymbol{I} \otimes \boldsymbol{I}] : \boldsymbol{\epsilon} = \mathbb{I}^{\mathrm{vol}} : \boldsymbol{\epsilon} \qquad (2.3.32)$$

index representation

$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \epsilon_{ij}^{\mathrm{vol}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad (2.3.33)$$

matrix representation of coordinates $[\epsilon_{ij}^{\mathrm{vol}}]$

$$[\epsilon_{ij}^{\mathrm{vol}}] = \frac{1}{3}\, e \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad e = \mathrm{tr}(\boldsymbol{\epsilon}) \qquad (2.3.34)$$

- incompressibility is characterized through $\mathrm{div}\, \boldsymbol{u} = 0$
- volumetric strain tensor $\boldsymbol{\epsilon}^{\mathrm{vol}}$ is a spherical second order tensor as $\boldsymbol{\epsilon}^{\mathrm{vol}} = \frac{1}{3} e \boldsymbol{I}$
- volumetric strain tensor $\boldsymbol{\epsilon}^{\mathrm{vol}}$ contains the volume changing, shape preserving part of the total strain tensor $\boldsymbol{\epsilon}$

Deviatoric deformation

deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ preserves the volume and contains the remaining part of the total strain tensor $\boldsymbol{\epsilon}$

deviatoric part $\boldsymbol{\epsilon}^{\text{dev}}$ of the strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon}^{\text{dev}} = \boldsymbol{\epsilon} - \boldsymbol{\epsilon}^{\text{vol}} = \boldsymbol{\epsilon} - \frac{1}{3}\left[\boldsymbol{\epsilon} : \boldsymbol{I}\right]\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\epsilon} \tag{2.3.35}$$

index representation

$$\boldsymbol{\epsilon}^{\text{dev}} = \epsilon_{ij}^{\text{dev}}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.36}$$

matrix representation of coordinates $[\epsilon_{ij}^{\text{dev}}]$

$$[\epsilon_{ij}^{\text{dev}}] = \frac{1}{3}\begin{bmatrix} 2\epsilon_{11} - \epsilon_{22} - \epsilon_{33} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & 2\epsilon_{22} - \epsilon_{11} - \epsilon_{33} & \epsilon_{13} \\ \epsilon_{31} & \epsilon_{32} & 2\epsilon_{33} - \epsilon_{11} - \epsilon_{22} \end{bmatrix} \tag{2.3.37}$$

trace of deviatoric strains $\text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right)$

$$\begin{aligned} \text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right) &= \tfrac{1}{3}\left[2\epsilon_{11} - \epsilon_{22} - \epsilon_{33}\right] \\ &+ \tfrac{1}{3}\left[2\epsilon_{22} - \epsilon_{11} - \epsilon_{33}\right] \\ &+ \tfrac{1}{3}\left[2\epsilon_{33} - \epsilon_{11} - \epsilon_{22}\right] = 0 \end{aligned} \tag{2.3.38}$$

• deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ is a traceless second order tensor as $\text{tr}\left(\boldsymbol{\epsilon}^{\text{dev}}\right) = 0$

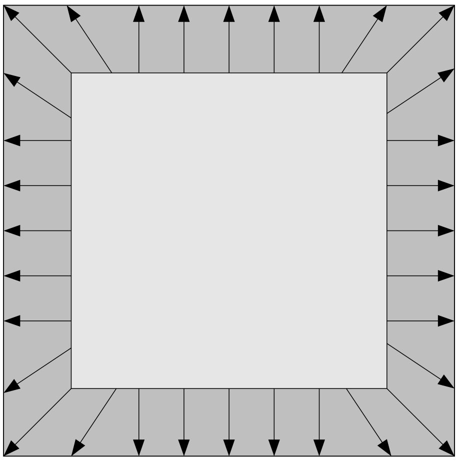
• deviatoric strain tensor $\boldsymbol{\epsilon}^{\text{dev}}$ contains the shape changing, volume preserving part of the total strain tensor $\boldsymbol{\epsilon}$

Volumetric–deviatoric decomposition of strain tensor

- examples of purely volumetric deformation

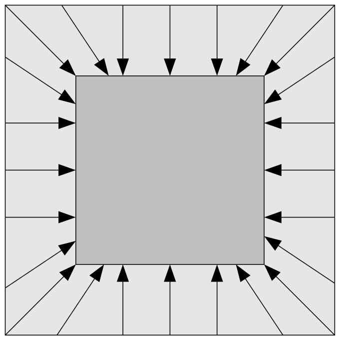
$$\boldsymbol{\epsilon}^{\mathrm{vol}} = \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\mathrm{vol}} : \boldsymbol{\epsilon} \qquad \mathrm{tr}(\boldsymbol{\epsilon}^{\mathrm{vol}}) = \mathrm{tr}(\boldsymbol{\epsilon}) \qquad (2.3.39)$$

expansion



$e > 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} = \mathbf{0}$

compression

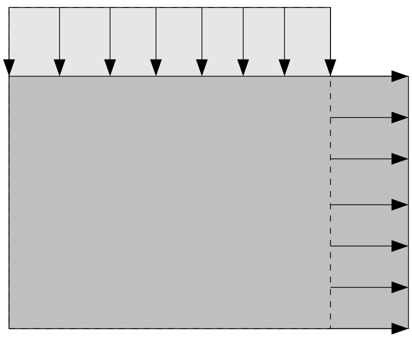


$e < 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} = \mathbf{0}$

- examples of purely deviatoric deformation

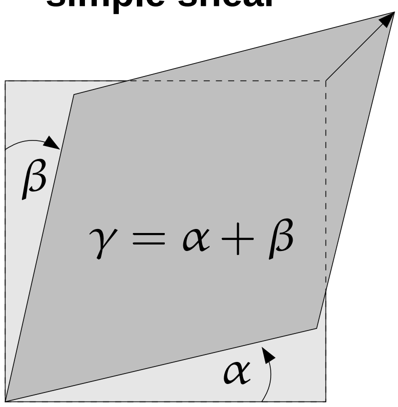
$$\boldsymbol{\epsilon}^{\mathrm{dev}} = \boldsymbol{\epsilon} - \frac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\mathrm{dev}} : \boldsymbol{\epsilon} \qquad \mathrm{tr}(\boldsymbol{\epsilon}^{\mathrm{dev}}) = 0 \qquad (2.3.40)$$

pure shear



$e = 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} \neq \mathbf{0}$

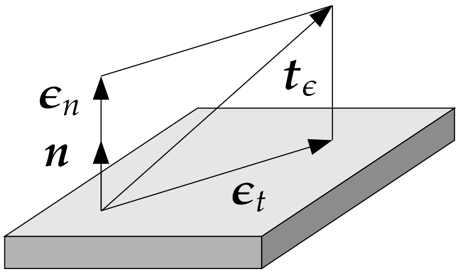
simple shear



$e = 0$ and $\boldsymbol{\epsilon}^{\mathrm{dev}} \neq \mathbf{0}$

2.3.5 Strain vector

assume we are interested in strain on a plane characterized through its normal $\boldsymbol{n}$, strain vector $\boldsymbol{t}_\epsilon$ acting on plane given through normal projection of strain tensor $\boldsymbol{\epsilon}$



$$\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon} \cdot \boldsymbol{n} \tag{2.3.41}$$

index representation

$$\begin{aligned} \boldsymbol{t}_\epsilon &= (\epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (n_k\, \boldsymbol{e}_k) \\ &= \epsilon_{ij}\, n_k \delta_{jk}\, \boldsymbol{e}_i = \epsilon_{ij}\, n_j\, \boldsymbol{e}_i = t_{\epsilon i}\, \boldsymbol{e}_i \end{aligned} \tag{2.3.42}$$

representation of coordinates $[t_{\epsilon i}]$

$$\begin{bmatrix} t_{\epsilon 1} \\ t_{\epsilon 2} \\ t_{\epsilon 3} \end{bmatrix} = \begin{bmatrix} \epsilon_{11}\, n_1 & + & \epsilon_{12}\, n_2 & + & \epsilon_{13}\, n_3 \\ \epsilon_{21}\, n_1 & + & \epsilon_{22}\, n_2 & + & \epsilon_{23}\, n_3 \\ \epsilon_{31}\, n_1 & + & \epsilon_{32}\, n_2 & + & \epsilon_{33}\, n_3 \end{bmatrix} \tag{2.3.43}$$

alternative interpretation: assume we are interested in strains along a particular material direction, i.e. the stretch of a fiber at $\boldsymbol{x} \in \mathcal{B}$ characterized through its normal $\boldsymbol{n}$ with $||\boldsymbol{n}|| = 1$

stretch as change of displacement vector $\boldsymbol{u}$ in the direction of $\boldsymbol{n}$ given through the Gateaux derivative §1.2.1.2

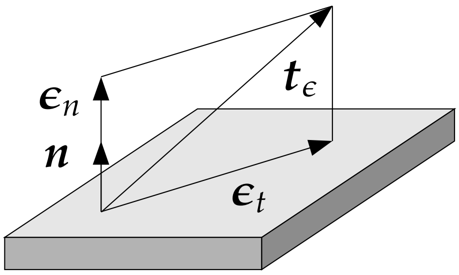
$$\begin{aligned} \mathrm{D}\, \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon} \boldsymbol{u}(\boldsymbol{x} + \epsilon\, \boldsymbol{n})|_{\epsilon=0} \\ &= \underbrace{\nabla \boldsymbol{u}(\boldsymbol{x} + \epsilon\, \boldsymbol{n})}_{\text{outer derviative}} \cdot \underbrace{\boldsymbol{n}}_{\text{inner derivative}} |_{\epsilon=0} = \nabla \boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{n} \end{aligned} \tag{2.3.44}$$

recall that $\nabla \boldsymbol{u} = \nabla^{\text{sym}} \boldsymbol{u} + \nabla^{\text{skw}} \boldsymbol{u} = \boldsymbol{\epsilon} + \boldsymbol{\omega}$ whereby rotation $\boldsymbol{\omega} = \nabla^{\text{skw}} \boldsymbol{u}$ does not induce strain, thus

$$\boldsymbol{t}_\epsilon = \nabla^{\text{sym}} \boldsymbol{u} \cdot \boldsymbol{n} = \epsilon \cdot \boldsymbol{n} \tag{2.3.45}$$

2.3.6 Normal–shear decomposition

assume we are interested in strain along a particular fiber characterized through its normal $\boldsymbol{n}$, stretch of fiber ϵ_n given through normal projection of strain vector $\boldsymbol{t}_\epsilon$



$$\epsilon_n = \boldsymbol{t}_\epsilon \cdot \boldsymbol{n} \tag{2.3.46}$$

alternative interpretation: stretch of a line element can be understood as the projection of change of displacement in the direction of $\boldsymbol{n}$ as $\mathrm{D}\boldsymbol{u} \cdot \boldsymbol{n} = \nabla \boldsymbol{u} \cdot \boldsymbol{n}$ onto the direction $\boldsymbol{n}$

$$\epsilon_n = \boldsymbol{n} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{n} = \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n}] \tag{2.3.47}$$

normal-shear (tangential) decomposition of strain vector $\boldsymbol{t}_\epsilon$

$$\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon}_n + \boldsymbol{\epsilon}_t \tag{2.3.48}$$

normal strain vector – stretch of fibers in direction of n

$$\boldsymbol{\epsilon}_n = \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n}]\, \boldsymbol{n} \tag{2.3.49}$$

shear (tangential) strain vector – sliding of fibers parallel to $\boldsymbol{n}$

$$\boldsymbol{\epsilon}_t = \boldsymbol{t}_\epsilon - \boldsymbol{\epsilon}_n = \boldsymbol{\epsilon} : [\mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \tag{2.3.50}$$

amount of sliding γ_n

$$\gamma_n = 2\,||\boldsymbol{\epsilon}_t|| = 2\sqrt{\boldsymbol{\epsilon}_t \cdot \boldsymbol{\epsilon}_t} = 2\sqrt{\boldsymbol{t}_\epsilon \cdot \boldsymbol{t}_\epsilon - \epsilon_n^2} \tag{2.3.51}$$

in general, i.e. for an arbitrary direction $\boldsymbol{n}$, we have normal and shear contributions to the strain vector, however, three particular directions $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ can be identified, for which $\boldsymbol{t}_\epsilon = \boldsymbol{\epsilon}_n$ and thus $\boldsymbol{\epsilon}_t = \boldsymbol{0}$, the corresponding $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ are called principal strain directions and $\{\epsilon_{n\, i}\}_{i=1,2,3} = \{\lambda_{\epsilon i}\}_{i=1,2,3}$ are the principal strains or stretches

2.3.7 Principal strains – stretches

assume strain tensor $\boldsymbol{\epsilon}$ to be known at $\boldsymbol{x} \in \mathcal{B}$, principal strains $\{\lambda_{\epsilon\, i}\}_{i=1,2,3}$ and principal strain directions $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ can be derived from solution of special eigenvalue problem according to §1.1.3

$$\boldsymbol{\epsilon} \cdot \boldsymbol{n}_{\epsilon\, i} = \lambda_{\epsilon\, i}\, \boldsymbol{n}_{\epsilon\, i} \qquad [\, \boldsymbol{\epsilon} - \lambda_{\epsilon\, i}\,] \cdot \boldsymbol{n}_{\epsilon\, i} = \boldsymbol{0} \tag{2.3.52}$$

solution

$$\det(\boldsymbol{\epsilon} - \lambda_{\epsilon}\, \boldsymbol{I}) = 0 \tag{2.3.53}$$

or in terms of roots of characteristic equation

$$\lambda_{\epsilon}^3 - I_{\epsilon}\, \lambda_{\epsilon}^2 + II_{\epsilon}\, \lambda_{\epsilon} - III_{\epsilon} = 0 \tag{2.3.54}$$

roots of characteristic equations in terms of principal invariants of $\boldsymbol{\epsilon}$

$$\begin{aligned} I_{\epsilon} &= \operatorname{tr}(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3} \\ II_{\epsilon} &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{\epsilon}) - \operatorname{tr}(\boldsymbol{\epsilon}^2)] &&= \lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 2} \\ III_{\epsilon} &= \det(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1}\, \lambda_{\epsilon 2}\, \lambda_{\epsilon 3} \end{aligned} \tag{2.3.55}$$

spectral representation of $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \sum_{i=1}^{3} \lambda_{\epsilon i}\, \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{2.3.56}$$

principal strains (stretches) $\lambda_{\epsilon i}$ are purely normal, no shear deformation (sliding) γ_n in principal directions, i.e. $\boldsymbol{t}_{\epsilon i} = \boldsymbol{\epsilon}_n = \lambda_{\epsilon i}\, \boldsymbol{n}_{\epsilon i}$ and $\boldsymbol{\epsilon}_t = \boldsymbol{0}$ thus $\gamma_n = 0$

due to symmetry of strains $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{t}}$, strain tensor possesses three real eigenvalues $\{\lambda_{\epsilon\, i}\}_{i=1,2,3}$, corresponding eigendirections $\{\boldsymbol{n}_{\epsilon\, i}\}_{i=1,2,3}$ are thus orthogonal $\boldsymbol{n}_{\epsilon i} \cdot \boldsymbol{n}_{\epsilon j} = \delta_{ij}$

2.3.8 Compatibility

until now, we have assumed the displacement field $\boldsymbol{u}(\boldsymbol{x},t)$ to be given, such that the strain field $\boldsymbol{\epsilon} = \nabla^{\text{sym}}\boldsymbol{u}$ could have been derived uniquely as partial derivative of $\boldsymbol{u}$ with respect to the position $\boldsymbol{x}$ at fixed time t
assume now, that for a given strain field $\boldsymbol{\epsilon}(\boldsymbol{x},t)$, we want to know whether these strains $\boldsymbol{\epsilon}$ are compatible with a continuous single–valued displacement field $\boldsymbol{u}$

symmetric second order incompatibility tensor

$$\boldsymbol{\eta} = \text{crl}(\text{crl}(\boldsymbol{\epsilon})) \tag{2.3.57}$$

index representation of incompatibility tensor

$$\boldsymbol{\eta} = \eta_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overset{3}{e}_{ikm}\ \epsilon_{kn,ml}\ \overset{3}{e}_{jln}\ \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \boldsymbol{0} \tag{2.3.58}$$

coordinate representation of compatibility condition

$$\epsilon_{kl,mn} + \epsilon_{mn,kl} - \epsilon_{ml,kn} - \epsilon_{kn,ml} = 0 \tag{2.3.59}$$

valid $\forall\ k,l,m,n$, thus 81 equations which are partly redundant, six independent conditions

St. Venant compatibility conditions

$$\begin{aligned}
\eta_{11} &= \epsilon_{22,33} + \epsilon_{33,22} - 2\,\epsilon_{23,32} &= 0\\
\eta_{22} &= \epsilon_{33,11} + \epsilon_{11,33} - 2\,\epsilon_{31,13} &= 0\\
\eta_{33} &= \epsilon_{11,22} + \epsilon_{22,11} - 2\,\epsilon_{12,21} &= 0\\
\eta_{12} &= \epsilon_{13,32} + \epsilon_{23,31} - \epsilon_{33,12} - \epsilon_{12,33} &= 0\\
\eta_{23} &= \epsilon_{21,13} + \epsilon_{31,12} - \epsilon_{11,23} - \epsilon_{23,11} &= 0\\
\eta_{31} &= \epsilon_{32,21} + \epsilon_{12,23} - \epsilon_{22,31} - \epsilon_{31,22} &= 0
\end{aligned} \tag{2.3.60}$$

incompatible displacement field, e.g. in dislocation theory

2.3.9 Special case of plane strain

dimensional reduction in case of plane strain with vanishing strains $\epsilon_{13} = \epsilon_{23} = \epsilon_{31} = \epsilon_{32} = \epsilon_{33} = 0$ in out of plane direction, e.g. in geomechanics

$$\boldsymbol{\epsilon} = \epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.61}$$

matrix representation of coordinates $[\epsilon_{ij}]$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.62}$$

2.3.10 Voigt representation of strain

three dimensional second order strain tensor $\boldsymbol{\epsilon}$

$$\boldsymbol{\epsilon} = \epsilon_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{2.3.63}$$

matrix representation of coordinates $[\epsilon_{ij}]$

$$[\epsilon_{ij}] = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \tag{2.3.64}$$

due to symmetry $[\epsilon_{ij}] = [\epsilon_{ji}]$ and thus $\epsilon_{12} = \epsilon_{21}$, $\epsilon_{23} = \epsilon_{32}$, $\epsilon_{31} = \epsilon_{13}$, strain tensor $\boldsymbol{\epsilon}$ contains only six independent components $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{23}, \epsilon_{31}$,it proves convenient to represent second order tensor $\boldsymbol{\epsilon}$ through a vector $\underline{\epsilon}$

$$\underline{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\,\epsilon_{12}, 2\,\epsilon_{23}, 2\,\epsilon_{31}]^{\mathrm{t}} \tag{2.3.65}$$

vector representation $\underline{\epsilon}$ of strain $\boldsymbol{\epsilon}$ in case of plane strain

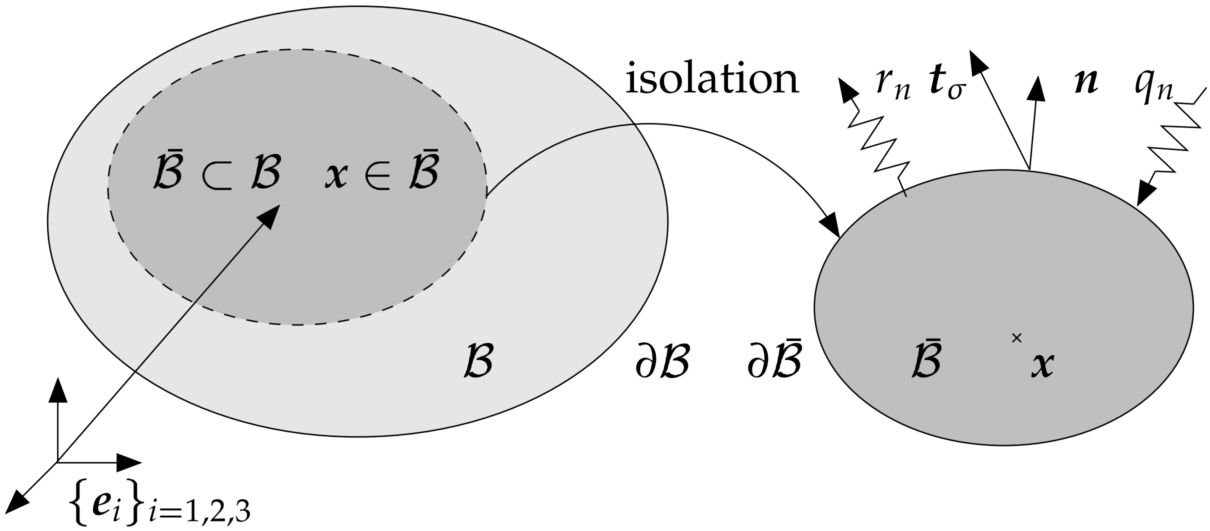
$$\underline{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\,\epsilon_{12}]^{\mathrm{t}} \tag{2.3.66}$$

3 Balance equations

3.1 Basic ideas

• until now:
kinematics, i.e. characterization of deformation of a material body $\mathcal{B}$ without studying its physical cause

• now:
balance equations, i.e. general statements that characterize the cause of cause of the motion of any body $\mathcal{B}$



Basic strategy

• isolation of an arbitrary subset $\bar{\mathcal{B}}$ of the body $\mathcal{B}$

• characterization of the influence of the remaining body $\mathcal{B}\backslash\bar{\mathcal{B}}$ on $\bar{\mathcal{B}}$ through phenomenological quantities, i.e. the contact mass flux r, the contact stress $\boldsymbol{t}_\sigma$, the contact heat flux q

• definition of basic physical quantities, i.e. the mass m, the linear momentum I, the moment of momentum D and the energy E of subset $\bar{\mathcal{B}}$

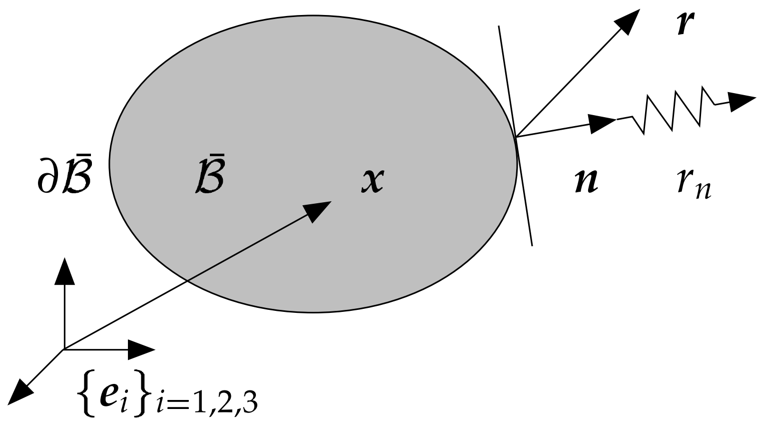
• postulate of balance of these quantities renders global balance equations for subset $\bar{\mathcal{B}}$

• localization of global balance equations renders local balance equations at point $\boldsymbol{x} \in \bar{\mathcal{B}}$

3.1.1 Concept of mass flux

the contact mass flux r_n at a point $\boldsymbol{x}$ is a scalar of the unit [mass/time/surface area]

the contact mass flux r_n characterizes the transport of matter normal to the tangent plane to an imaginary surface passing through this point with normal vector $\boldsymbol{n}$



definition of contact heat flux q_n in analogy to Cauchy's postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

Cauchy's postulate

$$r_n = r_n\,(\boldsymbol{x},\boldsymbol{n}) \tag{3.1.1}$$

Cauchy's lemma

$$r_n(\boldsymbol{x},\boldsymbol{n}) = -r_n(\boldsymbol{x},-\boldsymbol{n}) \tag{3.1.2}$$

Cauchy's theorem

the contact mass flux r_n can be expressed as linear function of the surface normal $\boldsymbol{n}$ and the mass flux vector $\boldsymbol{r}$

$$r_n = \boldsymbol{r}\cdot\boldsymbol{n} \tag{3.1.3}$$

Mass flux vector

the vector field $\boldsymbol{r}$ is called mass flux vector

$$\boldsymbol{r} = r_i\,\boldsymbol{e}_i \tag{3.1.4}$$

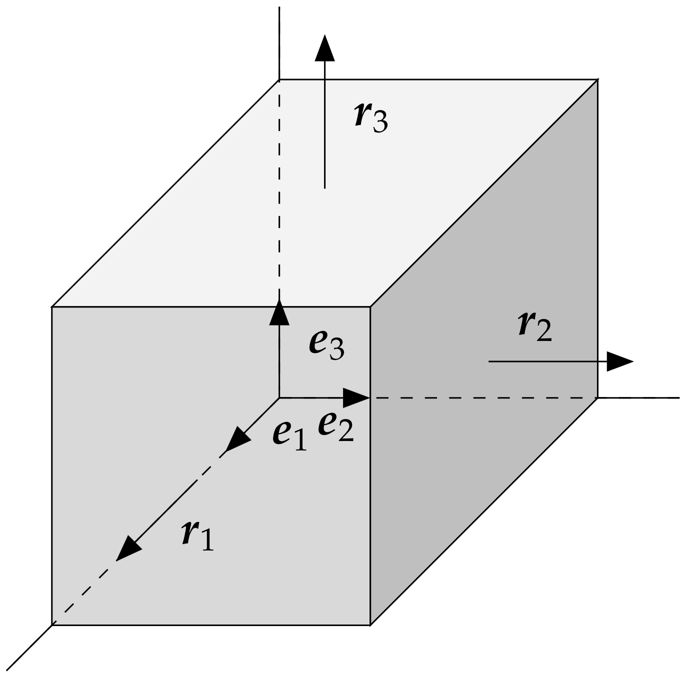
Cauchy's theorem

$$r_n = \boldsymbol{r} \cdot \boldsymbol{n} \tag{3.1.5}$$

index representation

$$r_n = (r_i\,\boldsymbol{e}_i) \cdot (n_j\,\boldsymbol{e}_j) = r_i\,n_j\,\delta_{ij} = r_i\,n_i \tag{3.1.6}$$

geometric interpretation



the coordinates r_i characterize the transport of matter through the planes parallel to the coordinate planes

in classical closed system continuum mechanics (here) the mass flux vector vanishes identically

examples of mass flux: transport of chemical reactants in chemomechanics or cell migration in biomechanics

3.1.2 Concept of stress

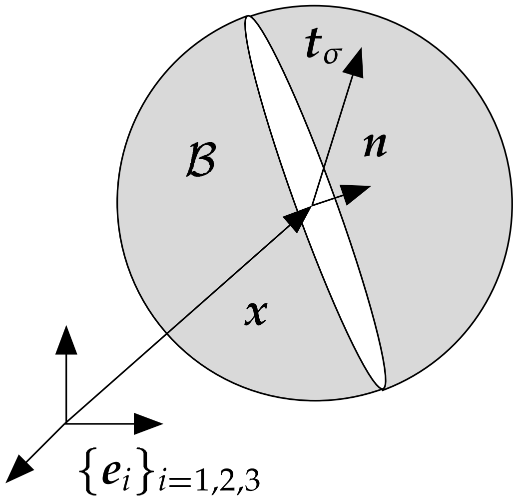
traction vector

$$\boldsymbol{t}_\sigma = \lim_{\Delta a \to 0} \frac{\Delta \boldsymbol{f}}{\Delta a} = \frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}a} \tag{3.1.7}$$

interpretation as surface force per unit surface area

Cauchy's postulate

the traction vector $\boldsymbol{t}_\sigma$ at a point $\boldsymbol{x}$ can be expressed exclu-



sively in terms of the point $\boldsymbol{x}$ and the normal $\boldsymbol{n}$ to the tangent plane to an imaginary surface passing through this point

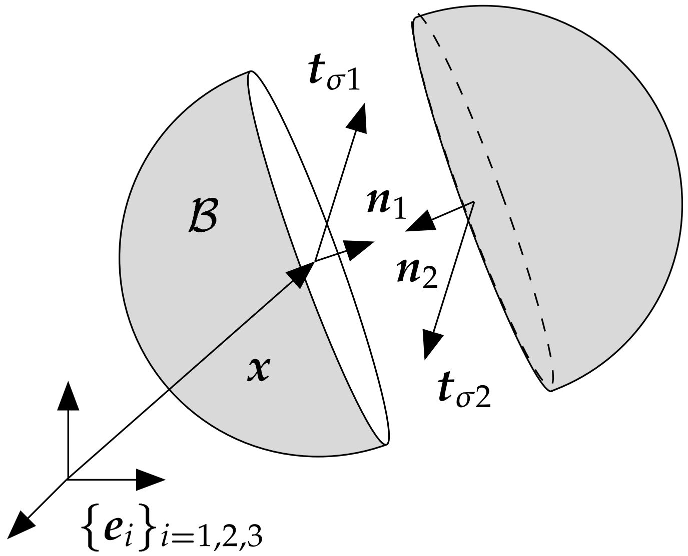
traction vector

$$\boldsymbol{t}_\sigma = \boldsymbol{t}_\sigma\,(\boldsymbol{x}, \boldsymbol{n}) \tag{3.1.8}$$

Cauchy's lemma

the traction vectors acting on opposite sides of a surface are equal in magnitude and opposite in sign

$$\boldsymbol{t}_{\sigma 1}(\boldsymbol{x}, \boldsymbol{n}_1) = -\boldsymbol{t}_{\sigma 2}(\boldsymbol{x}, \boldsymbol{n}_2) \tag{3.1.9}$$



generalization with $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ and $\boldsymbol{t}_\sigma = \boldsymbol{t}_{\sigma 1}$

$$\boldsymbol{t}_\sigma(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}_\sigma(\boldsymbol{x}, -\boldsymbol{n}) \tag{3.1.10}$$

Cauchy's theorem

the traction vector $\boldsymbol{t}_\sigma$ can be expressed as a linear map of the surface normal $\boldsymbol{n}$ mapped via the transposed stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} \tag{3.1.11}$$

accordingly with $\boldsymbol{n} = \boldsymbol{n}_1 = -\boldsymbol{n}_2$ and $\boldsymbol{t}_\sigma = \boldsymbol{t}_{\sigma 1}$

$$\begin{aligned} \boldsymbol{t}_{\sigma 1} &= \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_1 = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} = \boldsymbol{t}_\sigma \\ \boldsymbol{t}_{\sigma 2} &= \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_2 = - \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} = - \boldsymbol{t}_\sigma \end{aligned} \tag{3.1.12}$$

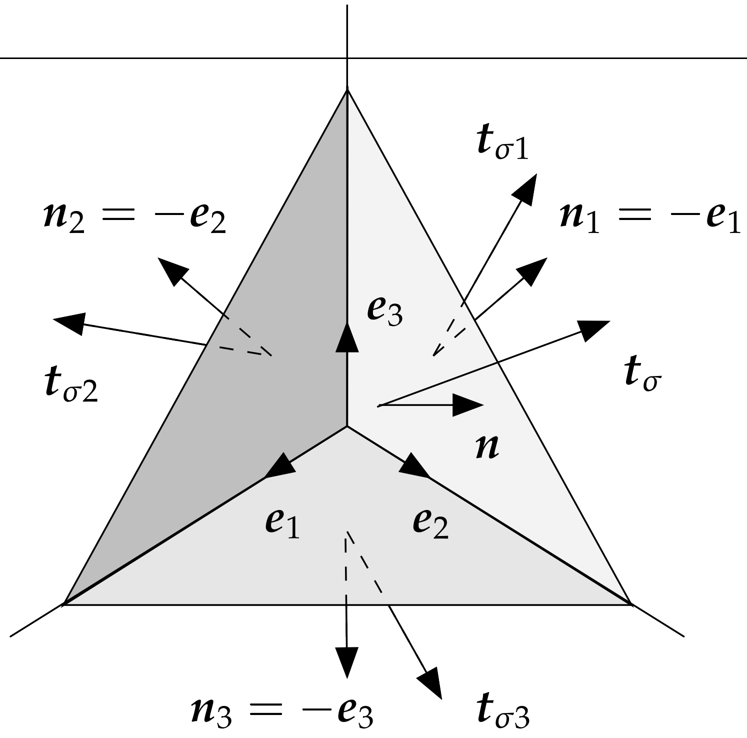
Cauchy tetraeder

balance of momentum (pointwise)

$$\boldsymbol{t}_\sigma(\boldsymbol{n})\, da = -\boldsymbol{t}_\sigma(\boldsymbol{n}_i)\, da_i = \boldsymbol{t}_\sigma(\boldsymbol{e}_i)\, da_i = \boldsymbol{t}_{\sigma i}\, da_i \tag{3.1.13}$$

surface theorem, area fractions from Gauss theorem

$$\boldsymbol{n} da = -\boldsymbol{n}_i da_i = \boldsymbol{e}_i da_i \qquad \frac{da_i}{da} = \boldsymbol{e}_i \cdot \boldsymbol{n} = \cos\angle(\boldsymbol{e}_i, \boldsymbol{n}) \tag{3.1.14}$$



traction vector as linear map of surface normal

$$\boldsymbol{t}_\sigma(\boldsymbol{n}) = \boldsymbol{t}_{\sigma i}\frac{da_i}{da} = \boldsymbol{t}_{\sigma i}\cos\angle(\boldsymbol{e_i},\boldsymbol{n}) = \boldsymbol{t}_{\sigma i}\,[\boldsymbol{e_i}\cdot\boldsymbol{n}] = [\boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e_i}]\cdot\boldsymbol{n} \tag{3.1.15}$$

compare $\boldsymbol{t}_\sigma(\boldsymbol{n}) = \boldsymbol{\sigma}^{\mathrm{t}}\cdot\boldsymbol{n}$
interpretation of second order stress tensor as $\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i$

Stress tensor

Cauchy stress (true stress)

$$\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i = \sigma_{ji}\boldsymbol{e}_j\otimes\boldsymbol{e}_i \qquad \boldsymbol{\sigma} = \boldsymbol{e}_i\otimes\boldsymbol{t}_{\sigma i} = \sigma_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j \tag{3.1.16}$$

Cauchy theorem

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\mathrm{t}}\cdot\boldsymbol{n} \tag{3.1.17}$$

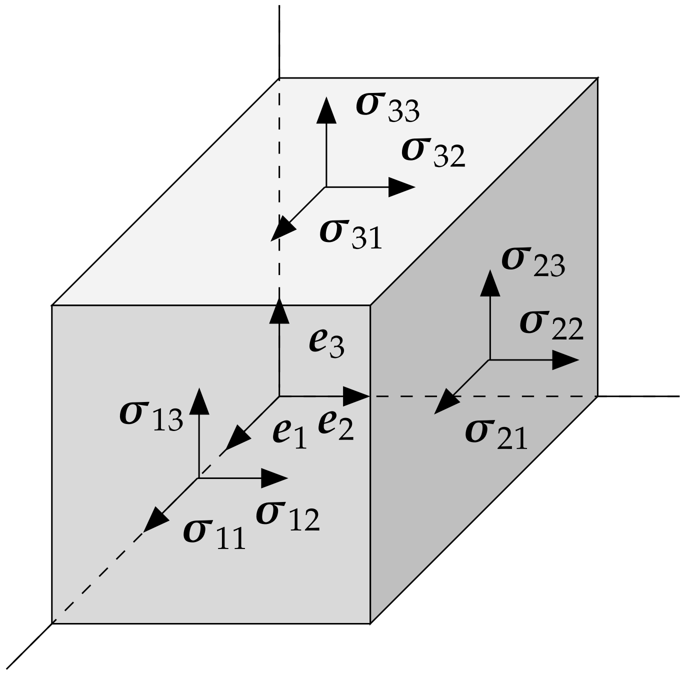
index representation

$$\boldsymbol{t}_\sigma = \sigma_{ji}\boldsymbol{e}_j\otimes\boldsymbol{e}_i\cdot n_k\boldsymbol{e}_k = \sigma_{ji}n_k\delta_{ik}\boldsymbol{e}_j = \sigma_{ji}n_i\boldsymbol{e}_j = t_j\boldsymbol{e}_j \tag{3.1.18}$$

matrix representation of tensor coordinates of σ_{ij}

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \boldsymbol{t}^{\mathrm{t}}_{\sigma 1} \\ \boldsymbol{t}^{\mathrm{t}}_{\sigma 2} \\ \boldsymbol{t}^{\mathrm{t}}_{\sigma 3} \end{bmatrix} \tag{3.1.19}$$

geometric interpretation



with traction vectors on surfaces

$$\begin{aligned} \boldsymbol{t}_{\sigma 1} &= [\ \sigma_{11}\ \ \sigma_{12}\ \ \sigma_{13}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_{\sigma 2} &= [\ \sigma_{21}\ \ \sigma_{22}\ \ \sigma_{23}\ ]^{\mathrm{t}} \\ \boldsymbol{t}_{\sigma 3} &= [\ \sigma_{31}\ \ \sigma_{32}\ \ \sigma_{33}\ ]^{\mathrm{t}} \end{aligned} \tag{3.1.20}$$

first index ... surface normal
second index ... direction (coordinate of traction vector)

diagonal entries ... normal stresses
non–diagonal entries .. shear stresses

3.1.2.1 Volumetric–deviatoric decomposition of stress tensor

in analogy to the strain tensor $\boldsymbol{\epsilon}$, the stress tensor $\boldsymbol{\sigma}$ can be additively decomposed into a volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ and a traceless deviatoric part $\boldsymbol{\sigma}^{\text{dev}}$

volumetric – deviatoric decomposition of stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{vol}} + \boldsymbol{\sigma}^{\text{dev}} \tag{3.1.21}$$

with volumetric and deviatoric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ and $\boldsymbol{\sigma}^{\text{dev}}$

$$\text{tr}(\boldsymbol{\sigma}^{\text{vol}}) = \text{tr}(\boldsymbol{\sigma}) \qquad \text{tr}(\boldsymbol{\sigma}^{\text{dev}}) = 0 \tag{3.1.22}$$

• volumetric second order tensor $\boldsymbol{\sigma}^{\text{vol}}$

$$\boldsymbol{\sigma}^{\text{vol}} = \frac{1}{3}[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{vol}} : \boldsymbol{\sigma} \tag{3.1.23}$$

upon double contraction volumetric fourth order unit tensor $\mathbb{I}^{\text{vol}}$ extracts volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ of stress tensor

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{3.1.24}$$

• deviatoric second order tensor $\boldsymbol{\sigma}^{\text{dev}}$

$$\boldsymbol{\sigma}^{\text{dev}} = \boldsymbol{\sigma} - \frac{1}{3}[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \mathbb{I}^{\text{dev}} : \boldsymbol{\sigma} \tag{3.1.25}$$

upon double contraction deviatoric fourth order unit tensor $\mathbb{I}^{\text{dev}}$ extracts deviatoric part of stress tensor

$$\begin{aligned} \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \left[\tfrac{1}{2}\,\delta_{ik}\delta_{jl} + \tfrac{1}{2}\,\delta_{il}\delta_{jk} - \tfrac{1}{3}\,\delta_{ij}\delta_{kl}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned} \tag{3.1.26}$$

Volumetric stress

volumetric part $\boldsymbol{\sigma}^{\text{vol}}$ of stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\text{vol}} = \tfrac{1}{3}\,[\boldsymbol{\sigma} : \boldsymbol{I}]\,\boldsymbol{I} = \tfrac{1}{3}\,[\boldsymbol{I} \otimes \boldsymbol{I}] : \boldsymbol{\sigma} = \mathbb{I}^{\text{vol}} : \boldsymbol{\sigma} \tag{3.1.27}$$

interpretation of trace as hydrostatic pressure

$$p = \tfrac{1}{3}\text{tr}(\boldsymbol{\sigma}) = \tfrac{1}{3}\,\boldsymbol{\sigma} : \boldsymbol{I} = \tfrac{1}{3}\,(\sigma_{11} + \sigma_{22} + \sigma_{33}) \tag{3.1.28}$$

index representation

$$\boldsymbol{\sigma}^{\text{vol}} = \sigma^{\text{vol}}_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.29}$$

matrix representation of coordinates $[\sigma^{\text{vol}}_{ij}]$

$$[\sigma^{\text{vol}}_{ij}] = p \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad p = \frac{1}{3}\,\text{tr}(\boldsymbol{\sigma}) \tag{3.1.30}$$

volumetric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ is a spherical second order tensor as $\boldsymbol{\sigma}^{\text{vol}} = p\,\boldsymbol{I}$

volumetric stress tensor $\boldsymbol{\sigma}^{\text{vol}}$ contains the hydrostatic pressure part of the total stress tensor $\boldsymbol{\sigma}$

Deviatoric stress

deviatoric stress tensor $\boldsymbol{\sigma}^{\rm dev}$ preserves the volume and containes the remaining part of the total stress tensor $\boldsymbol{\sigma}$

deviatoric part $\boldsymbol{\sigma}^{\rm dev}$ of the stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\rm dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\rm vol} = \boldsymbol{\sigma} - \tfrac{1}{3}\left[\boldsymbol{\sigma} : \boldsymbol{I}\right]\boldsymbol{I} = \mathbb{I}^{\rm dev} : \boldsymbol{\sigma} \tag{3.1.31}$$

index representation

$$\boldsymbol{\sigma}^{\rm dev} = \sigma^{\rm dev}_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.32}$$

matrix representation of coordinates $[\sigma^{\rm dev}_{ij}]$

$$[\sigma^{\rm dev}_{ij}] = \tfrac{1}{3}\begin{bmatrix} 2\sigma_{11} - \sigma_{22} - \sigma_{33} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & 2\sigma_{22} - \sigma_{11} - \sigma_{33} & \sigma_{13} \\ \sigma_{31} & \sigma_{32} & 2\sigma_{33} - \sigma_{11} - \sigma_{22} \end{bmatrix} \tag{3.1.33}$$

trace of deviatoric stresss $\mathrm{tr}\left(\boldsymbol{\sigma}^{\rm dev}\right)$

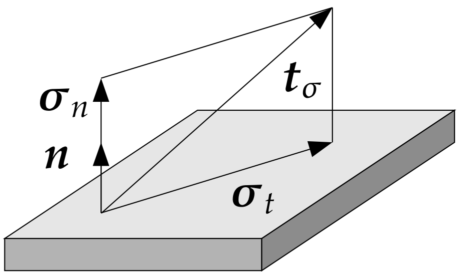
$$\begin{aligned} \mathrm{tr}\left(\boldsymbol{\sigma}^{\rm dev}\right) &= \tfrac{1}{3}\left[2\sigma_{11} - \sigma_{22} - \sigma_{33}\right] \\ &+ \tfrac{1}{3}\left[2\sigma_{22} - \sigma_{11} - \sigma_{33}\right] \\ &+ \tfrac{1}{3}\left[2\sigma_{33} - \sigma_{11} - \sigma_{22}\right] = 0 \end{aligned} \tag{3.1.34}$$

deviatoric stress tensor $\boldsymbol{\sigma}^{\rm dev}$ is a traceless second order tensor as $\mathrm{tr}\left(\boldsymbol{\sigma}^{\rm dev}\right) = 0$

deviatoric stress tensor $\boldsymbol{\sigma}^{\rm dev}$ contains the hydrostatic pressure free part of the total stress tensor $\boldsymbol{\sigma}$

3.1.2.2 Normal–shear decomposition

assume we are interested in the stress σ_n normal to a particular plane characterized through its normal $\boldsymbol{n}$, i.e. the normal projection of the stress vector $\boldsymbol{t}_\sigma$



$$\sigma_n = \boldsymbol{t}_\sigma \cdot \boldsymbol{n} = [\boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}] \cdot \boldsymbol{n} = \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\sigma}^{\mathrm{t}} : \boldsymbol{N} \quad (3.1.35)$$

normal–shear (tangential) decomposition of stress vector $\boldsymbol{t}_\sigma$

$$\boldsymbol{t}_\sigma = \boldsymbol{\sigma}_n + \boldsymbol{\sigma}_t \quad (3.1.36)$$

normal stress vector – stress in direction of $\boldsymbol{n}$

$$\boldsymbol{\sigma}_n = [\boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n}]]\, \boldsymbol{n} = \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \quad (3.1.37)$$

shear (tangential) stress vector – stress in the plane

$$\begin{aligned} \boldsymbol{\sigma}_t &= \boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n} - \boldsymbol{\sigma}^{\mathrm{t}} : [\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \\ &= \boldsymbol{\sigma}^{\mathrm{t}} : [\mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\sigma}^{\mathrm{t}} : \boldsymbol{T} \end{aligned} \quad (3.1.38)$$

amount of shear stress τ_n

$$||\tau_n||^2 = (\boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n) \cdot (\boldsymbol{t}_\sigma - \boldsymbol{\sigma}_n) = \boldsymbol{t}_\sigma \cdot \boldsymbol{t}_\sigma - 2\boldsymbol{t}_\sigma \cdot \boldsymbol{\sigma}_n + \sigma_n^2 \boldsymbol{n} \cdot \boldsymbol{n} \quad (3.1.39)$$

and thus

$$\tau_n = ||\boldsymbol{\sigma}_t|| = \sqrt{\boldsymbol{\sigma}_t \cdot \boldsymbol{\sigma}_t} = \sqrt{\boldsymbol{t}_\sigma \cdot \boldsymbol{t}_\sigma - \sigma_n^2} \quad (3.1.40)$$

in general, i.e. for an arbitrary direction $\boldsymbol{n}$, we have normal and shear contributions to the stress vector, however, three particular directions $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ can be identified, for which $\boldsymbol{t}_\sigma = \boldsymbol{\sigma}_n$ and thus $\boldsymbol{\sigma}_t = \boldsymbol{0}$, the corresponding $\{\boldsymbol{n}_{\sigma\, i}\}_{i=1,2,3}$ are called prinicpal stress directions and $\{\sigma_{n\, i}\}_{i=1,2,3} = \{\lambda_{\sigma i}\}_{i=1,2,3}$ are the principal stresses

3.1.2.3 Principal stresses

assume stress tensor $\boldsymbol{\sigma}^{\mathrm{t}}$ to be known at $\boldsymbol{x} \in \mathcal{B}$, principal stresses $\{\lambda_{\sigma\,i}\}_{i=1,2,3}$ and principal stress directions $\{\boldsymbol{n}_{\sigma\,i}\}_{i=1,2,3}$ can be derived from solution of special eigenvalue problem according to §1.1.3

$$\boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}_{\sigma\,i} = \lambda_{\sigma\,i}\,\boldsymbol{n}_{\sigma\,i} \qquad [\,\boldsymbol{\sigma}^{\mathrm{t}} - \lambda_{\sigma\,i}\,] \cdot \boldsymbol{n}_{\sigma\,i} = \boldsymbol{0} \tag{3.1.41}$$

solution

$$\det\left(\boldsymbol{\sigma}^{\mathrm{t}} - \lambda_{\sigma}\,\boldsymbol{I}\right) = 0 \tag{3.1.42}$$

or in terms of roots of characteristic equation

$$\lambda_{\sigma}^{3} - I_{\sigma}\,\lambda_{\sigma}^{2} + II_{\sigma}\,\lambda_{\sigma} - III_{\sigma} = 0 \tag{3.1.43}$$

roots of characteristic equation in terms of principal invariants of $\boldsymbol{\sigma}^{\mathrm{t}}$

$$\begin{aligned} I_{\sigma} &= \operatorname{tr}\left(\boldsymbol{\sigma}^{\mathrm{t}}\right) &&= \lambda_{\sigma 1} + \lambda_{\sigma 2} + \lambda_{\sigma 3} \\ II_{\sigma} &= \tfrac{1}{2}[\operatorname{tr}^{2}(\boldsymbol{\sigma}^{\mathrm{t}}) - \operatorname{tr}\left(\boldsymbol{\sigma}^{\mathrm{t}\,2}\right)] &&= \lambda_{\sigma 2}\lambda_{\sigma 3} + \lambda_{\sigma 3}\lambda_{\sigma 1} + \lambda_{\sigma 1}\lambda_{\sigma 2} \\ III_{\sigma} &= \det\left(\boldsymbol{\sigma}^{\mathrm{t}}\right) &&= \lambda_{\sigma 1}\,\lambda_{\sigma 2}\,\lambda_{\sigma 3} \end{aligned} \tag{3.1.44}$$

spectral representation of $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma}^{\mathrm{t}} = \sum_{i=1}^{3} \lambda_{\sigma i}\,\boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \tag{3.1.45}$$

principal stresses $\lambda_{\sigma i}$ are purely normal, no shear stress τ_{n} in principal directions, i.e. $\boldsymbol{t}_{\sigma i} = \boldsymbol{\sigma}_{n} = \lambda_{\sigma i}\,\boldsymbol{n}_{\sigma i}$ and $\boldsymbol{\sigma}_{t} = \boldsymbol{0}$ thus $\tau_{n} = 0$

due to symmetry of stresses $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$, stress tensor posseses three real eigenvalues $\{\lambda_{\sigma\,i}\}_{i=1,2,3}$, corresponding eigendirections $\{\boldsymbol{n}_{\sigma\,i}\}_{i=1,2,3}$ are thus orthogonal $\boldsymbol{n}_{\sigma i} \cdot \boldsymbol{n}_{\sigma j} = \delta_{ij}$

3.1.2.4 Special case of plane stress

dimensional reduction in case of plane stress with vanishing stresses $\sigma_{13} = \sigma_{23} = \sigma_{31} = \sigma_{32} = \sigma_{33} = 0$ in out of plane direction, e.g. for flat sheets

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.46}$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.1.47}$$

3.1.2.5 Voigt representation of stress

three dimensional second order stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \tag{3.1.48}$$

matrix representation of coordinates $[\sigma_{ij}]$

$$[\sigma_{ij}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{23} & \sigma_{33} \end{bmatrix} \tag{3.1.49}$$

due to symmetry $[\sigma_{ij}] = [\sigma_{ji}]$ and thus $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{31} = \sigma_{13}$, stress tensor $\boldsymbol{\sigma}$ contains only six independent components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}$,it proves convenient to represent second order tensor $\boldsymbol{\sigma}$ through a vector $\underline{\sigma}$

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^{\mathrm{t}} \tag{3.1.50}$$

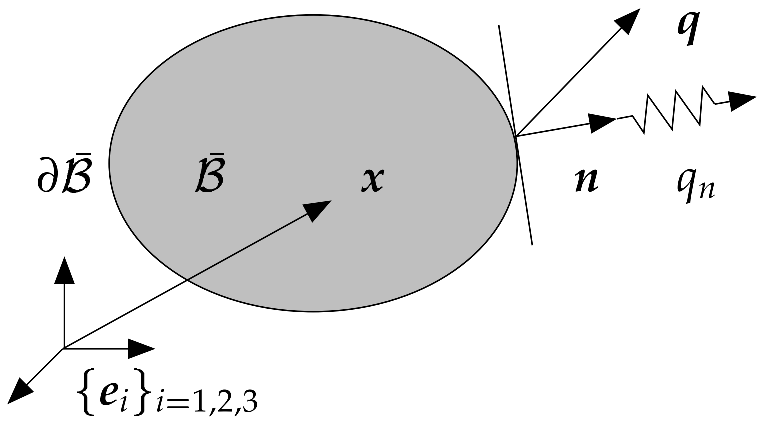
vector representation $\underline{\sigma}$ of stress $\boldsymbol{\sigma}$ in case of plane stress

$$\underline{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}]^{\mathrm{t}} \tag{3.1.51}$$

3.1.3 Concept of heat flux

the contact heat flux q_n at a point $\boldsymbol{x}$ is a scalar of the unit [energy/time/surface area]

the contact heat flux q_n characterizes the energy transport normal to the tangent plane to an imaginary surface passing through this point with normal vector $\boldsymbol{n}$



definition of contact heat flux q_n in analogy to Cauchy's postulate, lemma and theorem originally introduced for the momentum flux in §3.1.2

Cauchy's postulate

$$q_n = q_n(\boldsymbol{x}, \boldsymbol{n}) \tag{3.1.52}$$

Cauchy's lemma

$$q_n(\boldsymbol{x}, \boldsymbol{n}) = -q_n(\boldsymbol{x}, -\boldsymbol{n}) \tag{3.1.53}$$

Cauchy's theorem

the contact heat flux q_n can be expressed as linear function of the surface normal $\boldsymbol{n}$ and the heat flux vector $\boldsymbol{q}$

$$q_n = \boldsymbol{q} \cdot \boldsymbol{n} \tag{3.1.54}$$

Heat flux vector

the vector field $\boldsymbol{q}$ is called heat flux vector

$$\boldsymbol{q} = q_i \, \boldsymbol{e}_i \tag{3.1.55}$$

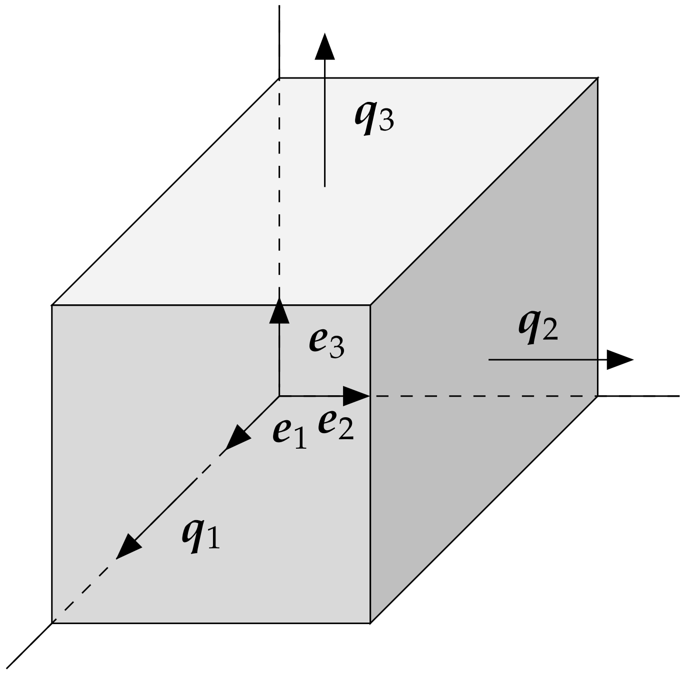
Cauchy's theorem

$$q_n = \boldsymbol{q} \cdot \boldsymbol{n} \tag{3.1.56}$$

index representation

$$q_n = (q_i \, \boldsymbol{e}_i) \cdot (n_j \, \boldsymbol{e}_j) = q_i \, n_j \, \delta_{ij} = q_i \, n_i \tag{3.1.57}$$

geometric interpretation



the coordinates q_i characterize the heat energy transport through the planes parallel to the coordinate planes

in continuum mechanics of adiabatic systems the heat flux vector vanishes identically

3.2 Balance of mass

total mass m of a body $\bar{\mathcal{B}}$

$$m := \int_{\bar{\mathcal{B}}} \mathrm{d}\, m \tag{3.2.1}$$

mass density

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{\mathrm{d}m}{\mathrm{d}v} \qquad \mathrm{d}\, m = \rho \,\mathrm{d}V \tag{3.2.2}$$

$$m = \int_{\bar{\mathcal{B}}} \rho \,\mathrm{d}V \tag{3.2.3}$$

mass exchange of body $\bar{\mathcal{B}}$ with environment m^{sur} and m^{vol}

$$m^{\text{sur}} := \int_{\partial\bar{\mathcal{B}}} r_n \,\mathrm{d}\, A \qquad m^{\text{vol}} := \int_{\bar{\mathcal{B}}} \mathcal{R} \,\mathrm{d}\, V \tag{3.2.4}$$

with contact mass flux $r_n = \boldsymbol{r} \cdot \boldsymbol{n}$ and mass source $\mathcal{R}$

3.2.1 Global form of balance of mass

"The time rate of change of the total mass m of a body $\bar{\mathcal{B}}$ is balanced with the mass exchange due to contact mass flux m^{sur} and the at-a-distance mass exchange m^{vol}."

$$\mathrm{D}_t m = m^{\text{sur}} + m^{\text{vol}} \tag{3.2.5}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \,\mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} r_n \,\mathrm{d}\, A + \int_{\bar{\mathcal{B}}} \mathcal{R} \,\mathrm{d}\, V \tag{3.2.6}$$

3.2.2 Local form of balance of mass

modification of rate term $\mathrm{D}_t\, m$

$$\mathrm{D}_t\, m = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho \,\mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t \rho \,\mathrm{d}V \tag{3.2.7}$$

modification of surface term m^{sur}

$$m^{\text{sur}} = \int_{\partial\bar{\mathcal{B}}} r_n \,\mathrm{d}\,A \overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{r}\cdot\boldsymbol{n} \,\mathrm{d}\,A \overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{r})\,\mathrm{d}\,V \tag{3.2.8}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t \rho \,\mathrm{d}V = \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{r})\,\mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \mathcal{R}\,\mathrm{d}\,V \tag{3.2.9}$$

for arbitrary bodies $\bar{\mathcal{B}} \to$ local form of mass balance

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$$\mathrm{D}_t \rho = \mathrm{div}\,(\boldsymbol{r}) + \mathcal{R} \tag{3.2.10}$$

3.2.3 Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), $\boldsymbol{r} = \boldsymbol{0}$ and $\mathcal{R} = 0$, such that the mass density ρ is constant in time

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$$\mathrm{D}_t\, \rho(\boldsymbol{x}, t) = 0 \qquad \rho = \rho(\boldsymbol{x}) = \text{const} \tag{3.2.11}$$

---

typically $\boldsymbol{r} \neq \boldsymbol{0}$ and $\mathcal{R} \neq 0$ only in bio– or chemomechanics

3.3 Balance of linear momentum

total momentum $\boldsymbol{p}$ of a body $\bar{\mathcal{B}}$

$$\boldsymbol{p} := \int_{\bar{\mathcal{B}}} \mathrm{D}_t \boldsymbol{u} \,\mathrm{d}\, m = \int_{\bar{\mathcal{B}}} \boldsymbol{v} \,\mathrm{d}\, m = \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{v} \,\mathrm{d} V \tag{3.3.1}$$

momentum exchange of body $\bar{\mathcal{B}}$ with environment through contact forces $\boldsymbol{f}^{\mathrm{sur}}$ and at-a-distance forces $\boldsymbol{f}^{\mathrm{vol}}$

$$\boldsymbol{f}^{\mathrm{sur}} := \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \,\mathrm{d}\, A \qquad \boldsymbol{f}^{\mathrm{vol}} := \int_{\bar{\mathcal{B}}} \boldsymbol{b} \,\mathrm{d}\, V \tag{3.3.2}$$

with contact/surface force $\boldsymbol{t}_\sigma = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}$ and volume force $\boldsymbol{b}$

3.3.1 Global form of balance of momentum

"The time rate of change of the total momentum $\boldsymbol{p}$ of a body $\bar{\mathcal{B}}$ is balanced with the momentum exchange due to contact momentum flux / surface force $\boldsymbol{f}^{\mathrm{sur}}$ and the at-a-distance momentum exchange / volume force $\boldsymbol{f}^{\mathrm{vol}}$."

$$\mathrm{D}_t \boldsymbol{p} = \boldsymbol{f}^{\mathrm{sur}} + \boldsymbol{f}^{\mathrm{vol}} \tag{3.3.3}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{v} \,\mathrm{d} V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \,\mathrm{d}\, A + \int_{\bar{\mathcal{B}}} \boldsymbol{b} \,\mathrm{d}\, V \tag{3.3.4}$$

3.3.2 Local form of balance of momentum

modification of rate term $\mathrm{D}_t\, \boldsymbol{p}$

$$\mathrm{D}_t\, \boldsymbol{p} = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{v} \mathrm{d} V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{v}) \mathrm{d} V \tag{3.3.5}$$

modification of surface term $\boldsymbol{f}^{\rm sur}$

$$\boldsymbol{f}^{\rm sur} = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{t}_\sigma \, \mathrm{d}\, A \overset{\rm Cauchy}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{\sigma}^{\rm t} \cdot \boldsymbol{n} \, \mathrm{d}\, A \overset{\rm Gauss}{=} \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{\sigma}^{\rm t})\, \mathrm{d}\, V \tag{3.3.6}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{v})\, \mathrm{d}V = \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{\sigma}^{\rm t})\, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \boldsymbol{b}\, \mathrm{d}\, V \tag{3.3.7}$$

for arbitrary bodies $\bar{\mathcal{B}} \to$ local form of momentum balance

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$$\mathrm{D}_t(\rho\, \boldsymbol{v}) = \mathrm{div}\,(\boldsymbol{\sigma}^{\rm t}) + \boldsymbol{b} \tag{3.3.8}$$

3.3.3 Reduction with lower order balance equations

balance equations are typically modified with the help of lower order balance equations

$$\mathrm{D}_t(\rho\, \boldsymbol{v}) = \boldsymbol{v}\, \mathrm{D}_t\, \rho + \rho\, \mathrm{D}_t\, \boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\rm t}) + \boldsymbol{b} \tag{3.3.9}$$

with balance of mass multiplied by velocity $\boldsymbol{v}$

$$\boldsymbol{v}\, \mathrm{D}_t\, \rho = \boldsymbol{v}\, \mathrm{div}\,(\boldsymbol{r}) + \boldsymbol{v}\, \mathcal{R}_0 = \mathrm{div}\,(\boldsymbol{v} \otimes \boldsymbol{r}) - \nabla \boldsymbol{v} \cdot \boldsymbol{r} + \boldsymbol{v}\, \mathcal{R}_0 \tag{3.3.10}$$

local momentum balance in reduced format

$$\rho\, \mathrm{D}_t \boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\rm t} - \nabla \boldsymbol{v} \otimes \boldsymbol{r}) + \boldsymbol{b} + \nabla \boldsymbol{v} \cdot \boldsymbol{r} - \boldsymbol{v}\, \mathcal{R}_0 \tag{3.3.11}$$

3.3.4 Classical continuum mechanics of closed systems

in classical closed system continuum mechanics (here), $\boldsymbol{r} = \mathbf{0}$ and $\mathcal{R} = 0$, such that the mass density ρ is constant in time, i.e. $\mathrm{D}_t(\rho\,\boldsymbol{v}) = \rho\,\mathrm{D}_t\boldsymbol{v}$ and thus

$$\rho\,\mathrm{D}_t\boldsymbol{v} = \mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{b} \tag{3.3.12}$$

the above equation is referred to as 'Cauchy's first equation of motion', Cauchy [1827]

3.4 Balance of angular momentum

total angular momentum $\boldsymbol{l}$ of a body $\bar{\mathcal{B}}$

$$\boldsymbol{l} := \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \mathrm{D}_t\boldsymbol{u}\,\mathrm{d}\,m = \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}\,m = \int_{\bar{\mathcal{B}}} \rho\,\boldsymbol{x} \times \boldsymbol{v}\,\mathrm{d}V \tag{3.4.1}$$

angular momentum exchange of body $\bar{\mathcal{B}}$ with environment through momentum from contact forces $\boldsymbol{l}^{\mathrm{sur}}$ and momentum from at-a-distance forces $\boldsymbol{l}^{\mathrm{vol}}$

$$\boldsymbol{l}^{\mathrm{sur}} := \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma\,\mathrm{d}\,A \qquad \boldsymbol{l}^{\mathrm{vol}} := \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b}\,\mathrm{d}\,V \tag{3.4.2}$$

with momentum due to contact/surface forces $\boldsymbol{x} \times \boldsymbol{t}_\sigma = \boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}$ and volume forces $\boldsymbol{x} \times \boldsymbol{b}$, assumption: no additional external torques

3.4.1 Global form of balance of angular momentum

"The time rate of change of the total angular momentum $\boldsymbol{l}$ of a body $\bar{\mathcal{B}}$ is balanced with the angular momentum exchange due to contact momentum flux / surface force $\boldsymbol{l}^{\text{sur}}$ and due to the at-a-distance momentum exchange / volume force $\boldsymbol{l}^{\text{vol}}$."

$$\mathrm{D}_t \boldsymbol{l} = \boldsymbol{l}^{\text{sur}} + \boldsymbol{l}^{\text{vol}} \tag{3.4.3}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{x} \times \boldsymbol{v} \,\mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma \,\mathrm{d}\,A + \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b} \,\mathrm{d}\,V \tag{3.4.4}$$

3.4.2 Local form of balance of angular momentum

modification of rate term $\mathrm{D}_t\, \boldsymbol{p}$

$$\mathrm{D}_t\, \boldsymbol{l} = \mathrm{D}_t \int_{\bar{\mathcal{B}}} \rho\, \boldsymbol{x} \times \boldsymbol{v} \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) \mathrm{d}V \tag{3.4.5}$$

modification of surface term $\boldsymbol{l}^{\text{sur}}$

$$\begin{aligned} \boldsymbol{l}^{\text{sur}} \quad &= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{t}_\sigma \,\mathrm{d}\,A \overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{\sigma}^{\text{t}} \cdot \boldsymbol{n} \,\mathrm{d}\,A \\ &\overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \operatorname{div}(\boldsymbol{x} \times \boldsymbol{\sigma}^{\text{t}}) \,\mathrm{d}\,V \\ &= \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \operatorname{div}(\boldsymbol{\sigma}^{\text{t}}) \,\mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \nabla \boldsymbol{x} \times \boldsymbol{\sigma}^{\text{t}} \,\mathrm{d}\,V \end{aligned} \tag{3.4.6}$$

and thus

$$\begin{aligned} \int_{\bar{\mathcal{B}}} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) \,\mathrm{d}V \;&=\; \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \operatorname{div}(\boldsymbol{\sigma}^{\text{t}}) \,\mathrm{d}\,V \\ &+ \int_{\bar{\mathcal{B}}} \boldsymbol{I} \times \boldsymbol{\sigma}^{\text{t}} \,\mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \boldsymbol{x} \times \boldsymbol{b} \,\mathrm{d}\,V \end{aligned} \tag{3.4.7}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of ang. mom. balance

$$\mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) = \boldsymbol{x} \times \operatorname{div}(\boldsymbol{\sigma}^{\text{t}}) + \boldsymbol{I} \times \boldsymbol{\sigma}^{\text{t}} + \boldsymbol{x} \times \boldsymbol{b} \tag{3.4.8}$$

3.4.3 Reduction with lower order balance equations

modification with the help of lower order balance equations

$$\begin{aligned} \mathrm{D}_t(\rho\, \boldsymbol{x} \times \boldsymbol{v}) &= \mathrm{D}_t\, \boldsymbol{x} \times (\rho\, \boldsymbol{v}) + \boldsymbol{x} \times \mathrm{D}_t\, (\rho\, \boldsymbol{v}) \\ &= \boldsymbol{x} \times \operatorname{div}(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{x} \times \boldsymbol{b} \end{aligned} \tag{3.4.9}$$

position vector $\boldsymbol{x}$ fixed, thus $\mathrm{D}_t\, \boldsymbol{x} = \boldsymbol{0}$, with balance of linear momentum multiplied by position $\boldsymbol{x}$

$$\boldsymbol{x} \times \mathrm{D}_t(\rho\, \boldsymbol{v}) = \boldsymbol{x} \times \operatorname{div}(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{x} \times \boldsymbol{b} \tag{3.4.10}$$

local angular momentum balance in reduced format

$$\boldsymbol{I} \times \boldsymbol{\sigma}^{\mathrm{t}} = \overset{3}{\boldsymbol{e}} : \boldsymbol{\sigma} = -2\, \operatorname{axl}(\boldsymbol{\sigma}) = \boldsymbol{0} \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}} \tag{3.4.11}$$

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the above equation is referred to as 'Cauchy's second equation of motion', Cauchy [1827]

in the absense of couple stresses, surface and body couples, the stress tensor is symmetric, $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$, else: micropolar / Cosserat continua

vector product of second order tensors $\boldsymbol{A} \times \boldsymbol{B} = \overset{3}{\boldsymbol{e}} : [\boldsymbol{A} \cdot \boldsymbol{B}^{\mathrm{t}}]$

3.5 Balance of energy

total energy E of a body $\bar{\mathcal{B}}$ as the sum of the kinetic energy K and the internal energy I

$$\begin{aligned} E &:= \int_{\bar{\mathcal{B}}} e \;\; \mathrm{d}V = \int_{\bar{\mathcal{B}}} k + i \, \mathrm{d}V = K + I \\ K &:= \int_{\bar{\mathcal{B}}} k \;\; \mathrm{d}V = \int_{\bar{\mathcal{B}}} \frac{1}{2} \rho \, \boldsymbol{v} \cdot \boldsymbol{v} \, \mathrm{d}V \\ I &:= \int_{\bar{\mathcal{B}}} i \;\; \mathrm{d}V \end{aligned} \tag{3.5.1}$$

energy exchange of body $\bar{\mathcal{B}}$ with environment e^{sur} and e^{vol}

$$\begin{aligned} E^{\mathrm{sur}} &:= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{t}_\sigma \, \mathrm{d}\,A - \int_{\partial\bar{\mathcal{B}}} q_n \, \mathrm{d}\,A \\ E^{\mathrm{vol}} &:= \int_{\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{b} \; \mathrm{d}\,V + \int_{\partial\bar{\mathcal{B}}} \mathcal{Q} \, \mathrm{d}\,V \end{aligned} \tag{3.5.2}$$

with contact heat flux $q_n = \boldsymbol{r} \cdot \boldsymbol{n}$ and heat source $\mathcal{Q}$

3.5.1 Global form of balance of energy

"The time rate of change of the total energy E of a body $\bar{\mathcal{B}}$ is balanced with the energy exchange due to contact energy flux E^{sur} and the at-a-distance energy exchange E^{vol}."

$$\mathrm{D}_t E = E^{\mathrm{sur}} + E^{\mathrm{vol}} \tag{3.5.3}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} e \, \mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{t}_\sigma - q_n \, \mathrm{d}A + \int_{\bar{\mathcal{B}}} \boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q} \mathrm{d}V \tag{3.5.4}$$

3.5.2 Local form of balance of energy

modification of rate term $\mathrm{D}_t\, E$

$$\mathrm{D}_t\, E = \mathrm{D}_t \int_{\bar{\mathcal{B}}} e\, \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t e\, \mathrm{d}V \tag{3.5.5}$$

modification of surface term E^{sur}

$$\begin{aligned} E^{\mathrm{sur}} &= \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v}\cdot\boldsymbol{t}_\sigma - q_n \,\mathrm{d}\,A \\ &\overset{\text{Cauchy}}{=} \int_{\partial\bar{\mathcal{B}}} \boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}\cdot\boldsymbol{n} - \boldsymbol{q}\cdot\boldsymbol{n} \,\mathrm{d}\,A \\ &\overset{\text{Gauss}}{=} \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}) + \mathrm{div}\,(-\boldsymbol{q})\,\mathrm{d}\,V \end{aligned} \tag{3.5.6}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t e\, \mathrm{d}V = \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q})\,\mathrm{d}\,V + \int_{\bar{\mathcal{B}}} \boldsymbol{v}\cdot\boldsymbol{b} + \mathcal{Q}\,\mathrm{d}\,V \tag{3.5.7}$$

for arbitrary bodies $\bar{\mathcal{B}} \rightarrow$ local form of energy balance

---

$$\mathrm{D}_t e = \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}) + \boldsymbol{v}\cdot\boldsymbol{b} + \mathcal{Q} \tag{3.5.8}$$

3.5.3 Reduction with lower order balance equations

modification with the help of lower order balance equations with

$$\begin{aligned} \mathrm{D}_t\, e &= \mathrm{D}_t\,(\,k + i\,) = v\,\mathrm{D}_t\,(\rho\,\boldsymbol{v}) + \mathrm{D}_t\, i \\ &= \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}) + \boldsymbol{v}\cdot\boldsymbol{b} + \mathcal{Q} \end{aligned} \tag{3.5.9}$$

with

$$\mathrm{D}_t\, k = \mathrm{D}_t(\tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v}) = \tfrac{1}{2}\mathrm{D}_t(\rho\boldsymbol{v})\cdot\boldsymbol{v} + \tfrac{1}{2}\boldsymbol{v}\cdot\mathrm{D}_t(\rho\boldsymbol{v}) = \boldsymbol{v}\cdot\mathrm{D}_t(\rho\boldsymbol{v})$$

(3.5.10)

with balance of momentum multiplied by velocity $\boldsymbol{v}$

$$\boldsymbol{v}\,\mathrm{D}_t\,(\rho\,\boldsymbol{v}) = \boldsymbol{v}\,\mathrm{div}\,(\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{v}\,\boldsymbol{b} = \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}) - \nabla\boldsymbol{v} : \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{v}\cdot\boldsymbol{b} \tag{3.5.11}$$

with stress power

$$\begin{aligned} \nabla\boldsymbol{v} : \boldsymbol{\sigma}^{\mathrm{t}} &= \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\boldsymbol{v} = \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\mathrm{D}_t\boldsymbol{\varphi} \\ &= \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\nabla\boldsymbol{\varphi} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\nabla^{\mathrm{sym}}\boldsymbol{\varphi} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} \end{aligned} \tag{3.5.12}$$

local energy balance in reduced format/balance of int. energy

---

$$\mathrm{D}_t\, i = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - \mathrm{div}\,(\boldsymbol{q}) + \mathcal{Q} \tag{3.5.13}$$

3.5.4 First law of thermodynamics

alternative definitions

$$\begin{aligned} \mathsf{p}^{\mathrm{ext}} &:= \mathrm{div}\,(\boldsymbol{v}\cdot\boldsymbol{\sigma}^{\mathrm{t}}) + \boldsymbol{v}\cdot\boldsymbol{b} && \text{... external mechanical power} \\ \mathsf{p}^{\mathrm{int}} &:= \boldsymbol{\sigma}^{\mathrm{t}} : \nabla\boldsymbol{v} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} && \text{... internal mechanical power} \\ \mathsf{q}^{\mathrm{ext}} &:= \mathrm{div}\,(-\boldsymbol{q}) + \mathcal{Q} && \text{... external thermal power} \end{aligned} \tag{3.5.14}$$

Balance of total energy / first law of thermodynamics

the rate of change of the total energy e, i.e. the sum of the kinetic energy k and the potential energy i is in equilibrium with the external mechanical power $\mathsf{p}^{\mathrm{ext}}$ and the external thermal power $\mathsf{q}^{\mathrm{ext}}$

$$\mathrm{D}_t\, e = \mathrm{D}_t\, k + \mathrm{D}_t\, i = \mathsf{p}^{\mathrm{ext}} + \mathsf{q}^{\mathrm{ext}} \tag{3.5.15}$$

the above equation is typically referred to as "principle of interconvertibility of heat and mechanical work" , Carnot [1832], Joule [1843], Duhem [1892]

first law of thermodynamics does not provide any information about the direction of a thermodynamic process

balance of kinetic energy

$$\mathrm{D}_t\, k = \mathsf{p}^{\mathrm{ext}} - \mathsf{p}^{\mathrm{int}} \tag{3.5.16}$$

the balance of kinetic energy is an alternative statement of the balance of linear momentum

balance of internal energy

$$\mathrm{D}_t\, i = \mathsf{q}^{\mathrm{ext}} + \mathsf{p}^{\mathrm{int}} \tag{3.5.17}$$

kinetic energy k and internal energy i are no conservation properties, they exchange the internal mechanical energy $\mathsf{p}^{\mathrm{int}}$

3.6 Balance of entropy

total entropy H of a body $\bar{\mathcal{B}}$

$$H := \int_{\bar{\mathcal{B}}} h \, \mathrm{d}V \tag{3.6.1}$$

entropy exchange of body $\bar{\mathcal{B}}$ with environment through H^{sur} and H^{vol}

$$H^{\text{sur}} := -\int_{\partial\bar{\mathcal{B}}} h_n \, \mathrm{d}\, A \qquad H^{\text{vol}} := \int_{\bar{\mathcal{B}}} \mathcal{H} \, \mathrm{d}\, V \tag{3.6.2}$$

with contact entropy flux $h_n = \boldsymbol{h} \cdot \boldsymbol{n}$ and entropy source $\mathcal{H}$

internal entropy production of body $\bar{\mathcal{B}}$ as H^{pro}

$$H^{\text{pro}} := \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}} \, \mathrm{d}\, V \geq 0 \tag{3.6.3}$$

whereby H^{pro} is strictly non-negative

3.6.1 Global form of balance of entropy

"The time rate of change of the total entropy H of a body $\bar{\mathcal{B}}$ is balanced with the energy exchange due to contact energy flux H^{sur}, the at-a-distance entropy exchange H^{vol} and the non–negative entropy production H^{pro} with $H^{\text{pro}} \geq 0$".

$$\mathrm{D}_t H = H^{\text{sur}} + H^{\text{vol}} + H^{\text{pro}} \tag{3.6.4}$$

and thus

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} h \mathrm{d}V = -\int_{\partial\bar{\mathcal{B}}} h_n \, \mathrm{d}A + \int_{\bar{\mathcal{B}}} \mathcal{H} \mathrm{d}V + \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}} \mathrm{d}V \tag{3.6.5}$$

3.6.2 Local form of balance of entropy

modification of rate term $\mathrm{D}_t\, H$

$$\mathrm{D}_t\, H = \mathrm{D}_t \int_{\bar{\mathcal{B}}} h\, \mathrm{d}V \overset{\bar{\mathcal{B}}\text{fixed}}{=} \int_{\bar{\mathcal{B}}} \mathrm{D}_t h\, \mathrm{d}V \tag{3.6.6}$$

modification of surface term H^{sur}

$$H^{\text{sur}} = - \int_{\partial\bar{\mathcal{B}}} h_n \mathrm{d}\, A \overset{\text{Cauchy}}{=} - \int_{\partial\bar{\mathcal{B}}} \boldsymbol{h} \cdot \boldsymbol{n} \mathrm{d}\, A \overset{\text{Gauss}}{=} - \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{h}) \mathrm{d}\, V \tag{3.6.7}$$

and thus

$$\int_{\bar{\mathcal{B}}} \mathrm{D}_t h\, \mathrm{d}V = - \int_{\bar{\mathcal{B}}} \mathrm{div}\,(\boldsymbol{h})\, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \mathcal{H}\, \mathrm{d}\, V + \int_{\bar{\mathcal{B}}} \mathsf{h}^{\text{pro}}\, \mathrm{d}\, V \tag{3.6.8}$$

for arbitrary bodies $\bar{\mathcal{B}} \to$ local form of entropy balance

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$$\mathrm{D}_t h = -\mathrm{div}\,(\boldsymbol{h}) + \mathcal{H} + \mathsf{h}^{\text{pro}} \tag{3.6.9}$$

3.6.3 Reduction with lower order balance equations

modification with the help of lower order balance equations assumption of existience of absolute temperature $theta \geq 0$ which renders the following constitutive assumption

$$\boldsymbol{h} = \frac{1}{\theta}\, \boldsymbol{q} \qquad \mathcal{H} = \frac{1}{\theta}\, \mathcal{Q} \tag{3.6.10}$$

such that

$$\begin{aligned} \mathrm{D}_t h &= - \qquad \mathrm{div}\left(\tfrac{1}{\theta}\boldsymbol{q}\right) \qquad + \tfrac{1}{\theta}\mathcal{Q} + \mathsf{h}^{\mathrm{pro}} \\ \theta\,\mathrm{D}_t h &= -\ \theta\,\tfrac{1}{\theta}\mathrm{div}\,(\boldsymbol{q}) - \boldsymbol{q}\cdot\theta\,\nabla\left(\tfrac{1}{\theta}\right) + \ \mathcal{Q} + \theta\,\mathsf{h}^{\mathrm{pro}} \\ \mathrm{D}_t(\theta\,h) &= -\,\theta\,\tfrac{1}{\theta}\mathrm{div}\,(\boldsymbol{q}) - \boldsymbol{q}\cdot\nabla\,\ln(\theta) + \ \mathcal{Q} + \theta\,\mathsf{h}^{\mathrm{pro}} - h\,\mathrm{D}_t\theta \end{aligned} \tag{3.6.11}$$

with balance of internal energy

$$\mathrm{D}_t\, i = \mathsf{q}^{\mathrm{ext}} + \mathsf{p}^{\mathrm{int}} = -\mathrm{div}\,(\boldsymbol{q}) + \mathcal{Q} + \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{3.6.12}$$

combination of balance of entropy and internal energy

$$\mathrm{D}_t\,(\,i - \theta\,h\,) = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - h\,\mathrm{D}_t\theta - \boldsymbol{q}\cdot\nabla\,\ln(\theta) - \theta\,\mathsf{h}^{\mathrm{pro}} \tag{3.6.13}$$

Legendre transform: free Helmholz energy

$$\psi = i - \theta\,h \tag{3.6.14}$$

and thus

$$\mathrm{D}_t\,\psi = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon} - h\,\mathrm{D}_t\theta - \boldsymbol{q}\cdot\nabla\,\ln(\theta) - \theta\,\mathsf{h}^{\mathrm{pro}} \tag{3.6.15}$$

3.6.4 Second law of thermodynamics

"The production of entropy $\mathsf{h}^{\mathrm{pro}}$ is non-negative."

$$\mathsf{h}^{\mathrm{pro}} \geq 0 \tag{3.6.16}$$

entropy is a measure of microscopic disorder of a system, positive entropy production gives a preferred direction to thermodynamic processes

Clausius: 'heat never flows from a colder to a warmer system'

introduction of dissipation (-rate) $\mathcal{D}$

$$\mathcal{D} := \theta\,\mathsf{h}^{\mathrm{pro}} \geq 0 \tag{3.6.17}$$

dissipation inequality

$$\mathcal{D} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t \boldsymbol{\epsilon} - h\, \mathrm{D}_t \theta - \mathrm{D}_t\, \psi - \boldsymbol{q} \cdot \nabla \ln(\theta) \geq 0 \qquad (3.6.18)$$

the above equation is referred to as 'Clausius–Duhem inequality', Clausius [1822-1888]

$$\begin{aligned} \mathcal{D} &= 0 \quad \text{... reversible process} \\ \mathcal{D} &> 0 \quad \text{... irreversible process} \end{aligned} \qquad (3.6.19)$$

conjugate pairs

$$\begin{array}{llll} \text{stress} & \boldsymbol{\sigma} & \text{vs.} & \text{strain} \quad \boldsymbol{\epsilon} \\ \text{entropy} & h & \text{vs.} & \text{temperature} \quad \theta \end{array} \qquad (3.6.20)$$

decomposition into local and conductive part

$$\mathcal{D} = \mathcal{D}^{\mathrm{loc}} + \mathcal{D}^{\mathrm{con}} \geq 0 \qquad (3.6.21)$$

local part / Clausius–Planck inequality

$$\mathcal{D}^{\mathrm{loc}} = \boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t \boldsymbol{\epsilon} - h\, \mathrm{D}_t \theta - \mathrm{D}_t\, \psi \geq 0 \qquad (3.6.22)$$

conductive part / Fourier inequality

$$\mathcal{D}^{\mathrm{con}} = -\boldsymbol{q} \cdot \nabla \ln(\theta) \geq 0 \qquad (3.6.23)$$

example: Fourier law of heat conduction

$$\boldsymbol{q} = -\boldsymbol{\kappa} \cdot \nabla \theta \qquad \text{isotropic} \qquad \boldsymbol{\kappa} = \kappa\, \boldsymbol{I} \qquad (3.6.24)$$

Fourier inequality a priori statisfied by construction

$$\nabla \theta \cdot \boldsymbol{\kappa} \cdot \nabla \theta \geq 0 \qquad (3.6.25)$$

3.7 Generic balance equation

any balance law can be expressed a general format

3.7.1 Global / integral format

$$\mathrm{D}_t\,\mathsf{A} = \mathsf{B} + \mathsf{C} + \Gamma \tag{3.7.1}$$

or alternatively

$$\mathrm{D}_t \int_{\bar{\mathcal{B}}} \mathsf{a}\,\mathrm{d}V = \int_{\partial\bar{\mathcal{B}}} \mathsf{b}\cdot\boldsymbol{n}\,\mathrm{d}A + \int_{\bar{\mathcal{B}}} \mathsf{c}\,\mathrm{d}V + \int_{\bar{\mathcal{B}}} \gamma\,\mathrm{d}V \tag{3.7.2}$$

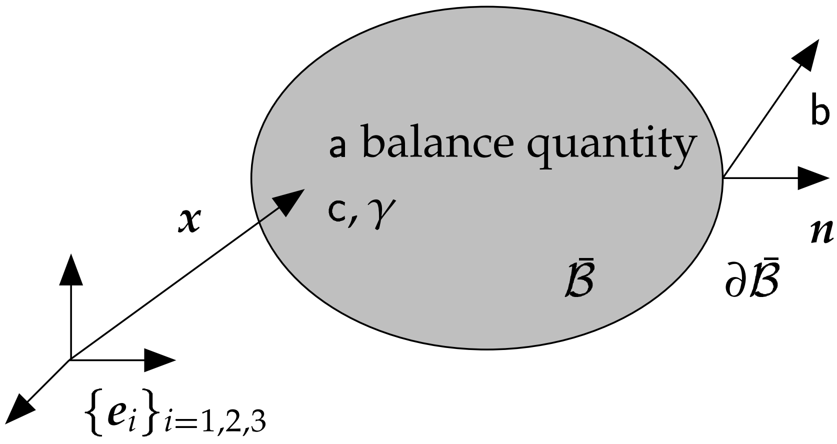
whereby

$$\begin{aligned} &\mathsf{A}\ \text{... balance quantity}\\ &\mathsf{B}\ \text{... surface transport through } \partial\bar{\mathcal{B}}\\ &\mathsf{C}\ \text{... volume source in } \bar{\mathcal{B}}\\ &\Gamma\ \text{... production in } \bar{\mathcal{B}} \end{aligned} \tag{3.7.3}$$

3.7.2 Local / differential format

local format of balance law follows from

- application of Gauss' theorem
- localization theorem, i.e. $\bar{\mathcal{B}}$ arbitrarily small



$$\mathrm{D}_t\, \mathsf{a} = \mathrm{div}\,(\mathsf{b}) + \mathsf{c} + \gamma \qquad (3.7.4)$$

	quantity a	flux b	source c	production γ
mass	ρ	$\boldsymbol{r}$	$\mathcal{R}$	–
lin. mom.	$\rho\boldsymbol{v}$	$\boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{b}$	–
ang. mom.	$\boldsymbol{x} \times \rho\boldsymbol{v}$	$\boldsymbol{x} \times \boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{x} \times \boldsymbol{b}$	–
energy	e	$\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}} - \boldsymbol{q}$	$\boldsymbol{v} \cdot \boldsymbol{b} + \mathcal{Q}$	–
kin. energy	k	$\boldsymbol{v} \cdot \boldsymbol{\sigma}^{\mathrm{t}}$	$\boldsymbol{v} \cdot \boldsymbol{b}$	$-\boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon}$
int. energy	i	$-\boldsymbol{q}$	$\mathcal{Q}$	$\boldsymbol{\sigma}^{\mathrm{t}} : \mathrm{D}_t\boldsymbol{\epsilon}$
entropy	h	$-\boldsymbol{h}$	$\mathcal{H}$	$\mathsf{h}^{\mathrm{pro}}$

Table 3.1: generic balance law

mass, linear momentum, angular momentum and total energy are conservation properties, while kinetic energy, internal energy and entropy are not, they possess a production term

3.8 Thermodynamic potentials

internal energy

$$i = i\,(\boldsymbol{\epsilon}, h, \ldots) \quad \rightarrow \quad \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} i \quad \text{and} \quad \theta = \mathrm{D}_h i \tag{3.8.1}$$

Legendre-Fenchel transform $h \rightarrow \theta$

$$\psi\,(\boldsymbol{\epsilon}, \theta) = \inf_h \left(i\,(\boldsymbol{\epsilon}, h) - \theta\, h\right) = i\,(\boldsymbol{\epsilon}, h(\theta)) - \theta\, h(\theta) \tag{3.8.2}$$

Helmholtz free energy

$$\psi = \psi\,(\boldsymbol{\epsilon}, \theta, \ldots) \quad \rightarrow \quad \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} \psi \quad \text{and} \quad h = -\mathrm{D}_\theta \psi \tag{3.8.3}$$

Legendre-Fenchel transform $\boldsymbol{\epsilon} \rightarrow \boldsymbol{\sigma}$

$$g\,(\boldsymbol{\sigma}, \theta) = \sup_{\boldsymbol{\epsilon}} \left(\boldsymbol{\sigma} : \boldsymbol{\epsilon} - \psi\,(\boldsymbol{\epsilon}, \theta)\right) = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{\sigma}) - \psi\,(\boldsymbol{\epsilon}(\boldsymbol{\sigma}), \theta) \tag{3.8.4}$$

Gibbs free energy

$$g = g\,(\boldsymbol{\sigma}, \theta, \ldots) \quad \rightarrow \quad \boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} g \quad \text{and} \quad h = \mathrm{D}_\theta g \tag{3.8.5}$$

Legendre-Fenchel transform $\theta \rightarrow h$

$$\eta\,(\boldsymbol{\sigma}, h) = \inf_\theta \left(g\,(\boldsymbol{\sigma}, \theta) - h\, \theta\right) = g\,(\boldsymbol{\sigma}, \theta(h)) - h\, \theta(h) \tag{3.8.6}$$

enthalpy

$$\eta = \eta\,(\boldsymbol{\epsilon}, h, \ldots) \quad \rightarrow \quad \boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} \eta \quad \text{and} \quad \theta = -\mathrm{D}_h \eta \tag{3.8.7}$$

Legendre-Fenchel transform $\boldsymbol{\sigma} \rightarrow \boldsymbol{\epsilon}$

$$i\,(\boldsymbol{\epsilon}, h) = \sup_{\boldsymbol{\sigma}} \left(\boldsymbol{\epsilon} : \boldsymbol{\sigma} - \eta\,(\boldsymbol{\sigma}, h)\right) = \boldsymbol{\epsilon} : \boldsymbol{\sigma}(\boldsymbol{\epsilon}) - \eta\,(\boldsymbol{\sigma}(\boldsymbol{\epsilon}), h) \tag{3.8.8}$$

4 Constitutive equations

motivation:

unknowns

• density ρ	1
• displacement $\boldsymbol{u}$	3
• temperature θ	1
• mass flux $\boldsymbol{r}$	3
• stress $\boldsymbol{\sigma}$	9
• heat flux $\boldsymbol{q}$	3
in total	$\underline{\underline{20}}$

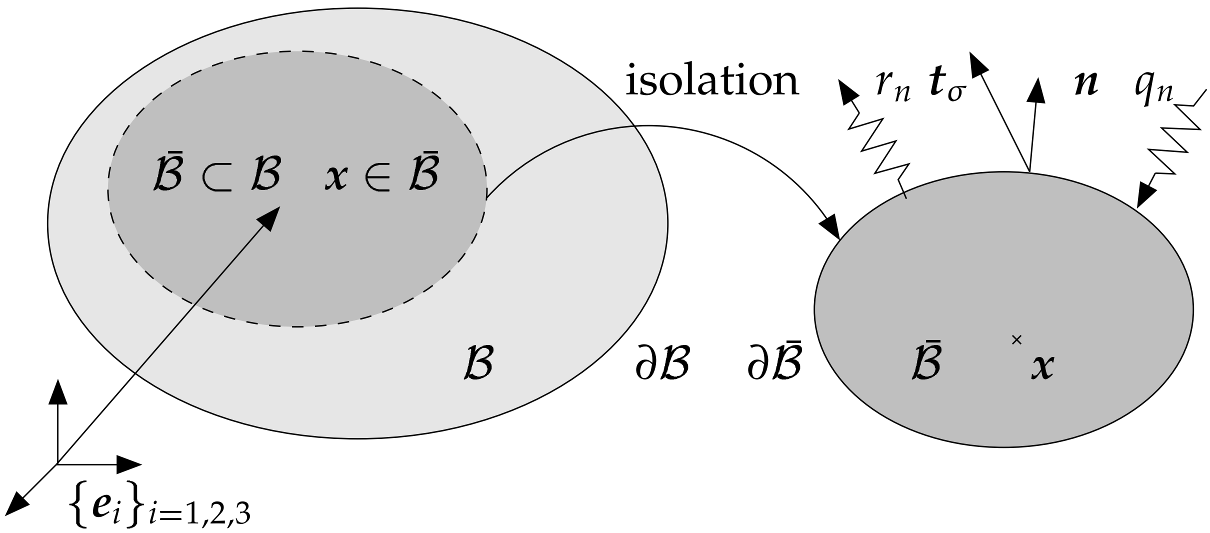
equations

• balance of mass	1
• balance of momentum	3
• balance of angular momentum	3
• balance of energy	1
in total	$\underline{\underline{8}}$

balance of unknowns vs. number of equations: 20-8=12 equations are missing, introduction of 12 material specific constitutive equations, i.e. equations for the mass flux $\boldsymbol{r}$ (3 eqns.), the symmetric stress $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}$ (6 eqns.) and the heat flux $\boldsymbol{q}$ (3 eqns.)

4.1 Linear constitutive equations

- for the mass flux $\boldsymbol{r}$ with $r_n = \boldsymbol{r} \cdot \boldsymbol{n}$
- for the momentum flux / stress $\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{\sigma}$ with $\boldsymbol{t}_n = \boldsymbol{\sigma}^{\mathrm{t}} \cdot \boldsymbol{n}$
- for the heat flux $\boldsymbol{q}$ with $q_n = \boldsymbol{q} \cdot \boldsymbol{n}$



in the simplest case, we could introduce ad hoc definitions of the mass flux, the momentum flux and the heat flux in terms of the spatial gradients of the density, the deformation and the temperature

4.1.1 Mass flux – Fick's law

linear relation between mass flux $\boldsymbol{r}$ (vector) and density gradient $\nabla\rho$ (vector) in terms of mass conduction coefficient $\boldsymbol{R}$ (second order tensor)

$$\boldsymbol{r} = \boldsymbol{R} \cdot \nabla\rho \tag{4.1.1}$$

index representation

$$r_i\,\boldsymbol{e}_i = [\,R_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,] \cdot [\,\rho_{,k}\boldsymbol{e}_k\,] = R_{ij}\boldsymbol{e}_i\delta_{jk}\rho_{,k} = R_{ij}\rho_{,j}\boldsymbol{e}_i \tag{4.1.2}$$

matrix representation of coordinates

$$[r_i] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} R_{11}\,\rho_{,1} & + & R_{12}\,\rho_{,2} & + & R_{13}\,\rho_{,3} \\ R_{21}\,\rho_{,1} & + & R_{22}\,\rho_{,2} & + & R_{23}\,\rho_{,3} \\ R_{31}\,\rho_{,1} & + & R_{32}\,\rho_{,2} & + & R_{33}\,\rho_{,3} \end{bmatrix} \tag{4.1.3}$$

special case of isotropie

$$\boldsymbol{R} = R\,\boldsymbol{I} \qquad \boldsymbol{r} = R\nabla\rho \qquad [r_i] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = R \begin{bmatrix} \rho_{,1} \\ \rho_{,2} \\ \rho_{,3} \end{bmatrix} \tag{4.1.4}$$

a linear relation between the flux of matter $\boldsymbol{r}$ and the gradient of concentrations $\nabla\rho$ is referred to as Fick's law

4.1.2 Momentum flux – Hook's law

linear relation between momentum flux $\boldsymbol{\sigma}$ (second order tensor) and displacement gradient $\nabla \boldsymbol{u}$ or rather $\boldsymbol{\epsilon} = \nabla^{\text{sym}} \boldsymbol{u}$ (second order tensor) in terms of elasticity tensor $\mathbb{E}$ (fourth order tensor)

$$\boldsymbol{\sigma} = \mathbb{E} : \nabla^{\text{sym}} \boldsymbol{u} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.1.5}$$

index representation

$$\begin{aligned} \sigma_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j &= [\, E_{ijkl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \,] \cdot [\, \epsilon_{mn} \boldsymbol{e}_m \otimes \boldsymbol{e}_n \,] \\ &= E_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \delta_{km} \delta_{ln} \epsilon_{mn} = E_{ijkl} \epsilon_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned} \tag{4.1.6}$$

special case of isotropie
i.e. identical Eigenbasis of stress & strain

$$\boldsymbol{\sigma} = \sum_{i_1}^{3} \lambda_{\sigma i} \boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \qquad \boldsymbol{\epsilon} = \sum_{i_1}^{3} \lambda_{\epsilon i} \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{4.1.7}$$

representation theorem for isotropic tensor–valued tensor-functions

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = f_1 \boldsymbol{I} + f_2 \boldsymbol{\epsilon} + f_3 \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \tag{4.1.8}$$

with $f_i = f_i(I_\epsilon, II_\epsilon, III_\epsilon)$ function of strain invariants

$$\begin{aligned} I_\epsilon &= \quad \text{tr}\,(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3} \\ II_\epsilon &= \tfrac{1}{2}[\text{tr}^2(\boldsymbol{\epsilon}) - \text{tr}\,(\boldsymbol{\epsilon}^2)] &&= \lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 2} \\ III_\epsilon &= \quad \det\,(\boldsymbol{\epsilon}) &&= \lambda_{\epsilon 1}\, \lambda_{\epsilon 2}\, \lambda_{\epsilon 3} \end{aligned} \tag{4.1.9}$$

a linear relation between the momentum flux $\boldsymbol{\sigma}$ and the strains $\boldsymbol{\epsilon}$ represents the generalized form of Hook's law

Hypoelasticity / Cauchy Elasticity

a hypoelastic / Cauchy elastic constitutive law can be represented in the following form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}) \tag{4.1.10}$$

- invertible relation between stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ (rates)
- possible dissipation of energy in closed strain circles

$$\oint \mathcal{D}^{\text{loc}}\,\mathrm{d}t = \oint \boldsymbol{\sigma} : \boldsymbol{\epsilon}\,\mathrm{d}t - \oint \mathrm{D}_t\psi\,\mathrm{d}t = \oint \boldsymbol{\sigma} : \boldsymbol{\epsilon}\,\mathrm{d}t \tag{4.1.11}$$

homogeneous strain path from $\boldsymbol{\epsilon}_{t_1}$ to $\boldsymbol{\epsilon}_{t_2}$

$$\boldsymbol{\epsilon}(\alpha) = [\,1-\alpha\,]\,\boldsymbol{\epsilon}_{t_1} + \alpha\,\boldsymbol{\epsilon}_{t_2} \qquad \mathrm{d}\boldsymbol{\epsilon} = [\,\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}\,]\mathrm{d}\alpha \tag{4.1.12}$$

stress work for linear elastic material

$$\begin{aligned} \int_{t_1}^{t_2} \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\epsilon} &= \int_0^1 \boldsymbol{\epsilon}(\alpha) : \mathbb{E} : [\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}]\mathrm{d}\alpha \\ &= \frac{1}{2}[\boldsymbol{\epsilon}_{t_2} + \boldsymbol{\epsilon}_{t_1}] : \mathbb{E} : [\boldsymbol{\epsilon}_{t_2} - \boldsymbol{\epsilon}_{t_1}] \end{aligned} \tag{4.1.13}$$

dissipation in isothermal closed strain cycle $t_1 \to t_2 \to t_3 \to t_1$

$$\oint \mathcal{D}^{\text{loc}}\,\mathrm{d}t = \boldsymbol{\epsilon}_{t_1} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_2} + \boldsymbol{\epsilon}_{t_2} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_3} + \boldsymbol{\epsilon}_{t_3} : \mathbb{E}^{\text{skw}} : \boldsymbol{\epsilon}_{t_1} \neq 0 \tag{4.1.14}$$

4.1.3 Heat flux – Fourier's law

linear relation between heat flux $\boldsymbol{q}$ (vector) and temperature gradient $\nabla\theta$ (vector) in terms of heat conduction coefficient $\boldsymbol{\kappa}$ (second order tensor)

$$\boldsymbol{q} = \boldsymbol{\kappa} \cdot \nabla\theta \tag{4.1.15}$$

index representation

$$q_i\,\boldsymbol{e}_i = [\,\kappa_{ij}\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\,] \cdot [\,\theta_{,k}\boldsymbol{e}_k\,] = \kappa_{ij}\boldsymbol{e}_i\delta_{jk}\theta_{,k} = \kappa_{ij}\theta_{,j}\boldsymbol{e}_i \tag{4.1.16}$$

matrix representation of coordinates

$$[q_i] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \kappa_{11}\,\theta_{,1} & + & \kappa_{12}\,\theta_{,2} & + & \kappa_{13}\,\theta_{,3} \\ \kappa_{21}\,\theta_{,1} & + & \kappa_{22}\,\theta_{,2} & + & \kappa_{23}\,\theta_{,3} \\ \kappa_{31}\,\theta_{,1} & + & \kappa_{32}\,\theta_{,2} & + & \kappa_{33}\,\theta_{,3} \end{bmatrix} \tag{4.1.17}$$

special case of isotropie

$$\boldsymbol{\kappa} = \kappa\,\boldsymbol{I} \qquad \boldsymbol{q} = \kappa\nabla\theta \qquad [q_i] = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \kappa \begin{bmatrix} \theta_{,1} \\ \theta_{,2} \\ \theta_{,3} \end{bmatrix} \tag{4.1.18}$$

a linear relation between the heat flux vector $\boldsymbol{q}$ and the temperature gradient $\nabla\theta$ is referred to as Fourier's law which goes back to Fourier [1822]

4.2 Hyperelasticity

4.2.1 Specific stored energy

a hyperelastic / Green elastic constitutive law can be represented in the following form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\epsilon}) \qquad \text{and} \qquad \mathcal{D}^{\text{loc}} = \mathcal{W} - \mathrm{D}_t\psi = 0 \tag{4.2.1}$$

• invertible relation between stress $\boldsymbol{\sigma}$ and strain $\boldsymbol{\epsilon}$ (and stress and strain rates $\mathrm{D}_t\boldsymbol{\sigma}$ and $\mathrm{D}_t\boldsymbol{\epsilon}$) based on a potential
• potential corresponds to elastically stored specific energy
• by construction no dissipation of energy in closed strain circles

stress power $\mathcal{W}$

$$\mathcal{W} = \boldsymbol{\sigma} : \mathrm{D}_t\boldsymbol{\epsilon} \doteq \mathrm{D}_t\psi \tag{4.2.2}$$

ensuring $\mathcal{D}^{\text{loc}} = \mathcal{W} - \mathrm{D}_t\psi = 0$ by construction, thus

$$\psi = \psi(\boldsymbol{\epsilon}) \qquad \text{and} \qquad \mathrm{D}_t\psi = \mathrm{D}_{\boldsymbol{\epsilon}}\psi : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.2.3}$$

specific stored energy W as path independent integral of stress power $\mathcal{W}$

$$W(\boldsymbol{\epsilon}) = \psi(\boldsymbol{\epsilon}) \tag{4.2.4}$$

with

$$W(\boldsymbol{\epsilon}_{t2}) - W(\boldsymbol{\epsilon}_{t1}) = \int_{t_1}^{t_2} \mathrm{D}_t W \mathrm{d}t = \int_{t_1}^{t_2} \mathcal{W} \mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\epsilon} \tag{4.2.5}$$

generic hyperelastic / Green elastic constitutive law

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W \qquad \text{with} \qquad W = W(\boldsymbol{\epsilon}) \tag{4.2.6}$$

- path independent $\qquad W(\boldsymbol{\epsilon}_{t2}) - W(\boldsymbol{\epsilon}_{t1}) = \int_{t_1}^{t_2} \mathrm{d}\, W$
- no dissipation $\qquad \oint \mathrm{d}\, W = 0$
- symmetric $\qquad \dfrac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\epsilon} \otimes \mathrm{D}\boldsymbol{\epsilon}}$

relation between stress rates and strain rates defines continuum tangent stiffness (fourth order tensor) $\mathbb{E}^{\tan}$

$$\mathrm{D}_t \boldsymbol{\sigma} = \mathbb{E}^{\tan} : \mathrm{D}_t \boldsymbol{\epsilon} \tag{4.2.7}$$

fourth order tangent stiffness / elastic material tangent

$$\mathbb{E}^{\tan} = \frac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\epsilon} \otimes \mathrm{D}\boldsymbol{\epsilon}} = E_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{4.2.8}$$

minor and major symmetries: reduction from $3^4 = 81$ to $6^2 = 36$ to 21 coefficients

$$E_{ijkl} = E_{jikl} = E_{jilk} = E_{ijlk} \qquad \text{and} \qquad E_{ijkl} = E_{klij} \tag{4.2.9}$$

4.2.2 Specific complementary energy

specific stored energy

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W \qquad \text{with} \qquad W = W(\boldsymbol{\epsilon}) \tag{4.2.10}$$

Legendre-Fenchel tranform $\boldsymbol{\sigma} \to \boldsymbol{\epsilon}$

$$W^*(\boldsymbol{\sigma}) = \sup_{\boldsymbol{\epsilon}}(\boldsymbol{\sigma} : \boldsymbol{\epsilon} - W(\boldsymbol{\epsilon})) \tag{4.2.11}$$

specific complementary stored energy

$$W^* = W^*(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{\sigma}) - W(\boldsymbol{\epsilon}(\boldsymbol{\sigma})) \tag{4.2.12}$$

general hyperelastic constitutive law

$$\boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} W^* \qquad \text{with} \qquad W^* = W^*(\boldsymbol{\sigma}) \tag{4.2.13}$$

---

relation between strain rates and stress rates defines continuum tangent compliance (fourth order tensor)$\mathbb{C}^{\tan}$

$$\mathrm{D}_t \boldsymbol{\epsilon} = \mathbb{C}^{\tan} : \mathrm{D}_t \boldsymbol{\sigma} \qquad \text{with} \qquad \mathbb{C}^{\tan} = \mathbb{E}^{\tan\,-1} \tag{4.2.14}$$

fourth order tangent compliance

$$\mathbb{C}^{\tan} = \frac{\mathrm{D}^2 W}{\mathrm{D}\boldsymbol{\sigma} \otimes \mathrm{D}\boldsymbol{\sigma}} = C_{ijkl} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{4.2.15}$$

minor and major symmetries: reduction from $3^4 = 81$ to $6^2 = 36$ to 21 coefficients

$$C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk} \qquad \text{and} \qquad C_{ijkl} = C_{klij} \tag{4.2.16}$$

4.3 Isotropic hyperelasticity

4.3.1 Specific stored energy

isotropy: identical eigenbasis of stress and strain

$$\boldsymbol{\sigma} = \sum_{i_1}^{3} \lambda_{\sigma i} \boldsymbol{n}_{\sigma i} \otimes \boldsymbol{n}_{\sigma i} \qquad \boldsymbol{\epsilon} = \sum_{i_1}^{3} \lambda_{\epsilon i} \boldsymbol{n}_{\epsilon i} \otimes \boldsymbol{n}_{\epsilon i} \tag{4.3.1}$$

representation theorem for isotropic tensor–valued tensor functions

---

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon}) = f_1 \boldsymbol{I} + f_2 \boldsymbol{\epsilon} + f_3 \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} \tag{4.3.2}$$

---

with $f_i = f_i(I_\epsilon, II_\epsilon, III_\epsilon)$ function of strain invariants

$$\begin{aligned} I_\epsilon &= \operatorname{tr}(\boldsymbol{\epsilon}) \\ II_\epsilon &= \tfrac{1}{2}[\operatorname{tr}^2(\boldsymbol{\epsilon}) - \operatorname{tr}(\boldsymbol{\epsilon}^2)] \\ III_\epsilon &= \det(\boldsymbol{\epsilon}) \end{aligned} \tag{4.3.3}$$

representation theorem for isotropic scalar–valued tensor functions

---

$$W(\boldsymbol{\epsilon}) = W(I_\epsilon, II_\epsilon, III_\epsilon) \tag{4.3.4}$$

---

with $W(\boldsymbol{\epsilon}) = W(\boldsymbol{Q} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{Q}^{\mathrm{t}}) \; \forall \, \boldsymbol{Q} \in SO(3)$

stress for hyperelastic material

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = \frac{\mathrm{D}W}{\mathrm{D}I_\epsilon} \frac{\mathrm{D}I_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}II_\epsilon} \frac{\mathrm{D}II_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}III_\epsilon} \frac{\mathrm{D}III_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \tag{4.3.5}$$

with derivatives of invariants I_ϵ, II_ϵ, III_ϵ with respect to second order tensor $\boldsymbol{\epsilon}$

$$\begin{aligned} \mathrm{D}_{\boldsymbol{\epsilon}}\, I_\epsilon &= \boldsymbol{I} \\ \mathrm{D}_{\boldsymbol{\epsilon}}\, II_\epsilon &= -\boldsymbol{\epsilon} + I_\epsilon\, \boldsymbol{I} \\ \mathrm{D}_{\boldsymbol{\epsilon}}\, III_\epsilon &= = III_\epsilon\, \boldsymbol{\epsilon}^{-\mathrm{t}} = \boldsymbol{\epsilon}^2 - I_\epsilon \boldsymbol{\epsilon} + II_\epsilon \boldsymbol{I} \end{aligned} \tag{4.3.6}$$

general representation of stress

$$\boldsymbol{\sigma} = \mathrm{D}_{I_\epsilon} W\, \boldsymbol{I} + \mathrm{D}_{II_\epsilon} W\, [-\boldsymbol{\epsilon} + I_\epsilon\, \boldsymbol{I}] + \mathrm{D}_{III_\epsilon} W\, [\boldsymbol{\epsilon}^2 - I_\epsilon \boldsymbol{\epsilon} + II_\epsilon \boldsymbol{I}] \tag{4.3.7}$$

comparison of coefficients

$$\begin{aligned} f_1 &= \mathrm{D}_{I_\epsilon} W + I_\epsilon\, \mathrm{D}_{II_\epsilon} W + II_\epsilon\, \mathrm{D}_{III_\epsilon} W \\ f_2 &= -\ \mathrm{D}_{II_\epsilon} W - I_\epsilon\, \mathrm{D}_{III_\epsilon} W \\ f_3 &= \mathrm{D}_{III_\epsilon} W \end{aligned} \tag{4.3.8}$$

assumption of linearity (quadratic term vanishes), two Lamé constants λ and μ

$$f_1 = I_\epsilon \lambda = [\boldsymbol{\epsilon} : \boldsymbol{I}]\lambda \qquad f_2 = 2\,\mu \qquad f_3 = 0 \tag{4.3.9}$$

specific stored energy (quadratic in strains)

$$W = \frac{1}{2}\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} = \frac{1}{2}\lambda\, [\boldsymbol{\epsilon} : \boldsymbol{I}]^2 + \mu[\boldsymbol{\epsilon}^2 : \boldsymbol{I}] \tag{4.3.10}$$

stress tensor (linear in strains)

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = \mathbb{E} : \boldsymbol{\epsilon} = f_1 \boldsymbol{I} + f_2 \boldsymbol{\epsilon} = \lambda[\, \boldsymbol{\epsilon} : \boldsymbol{I}\,]\, \boldsymbol{I} + 2\,\mu\, \boldsymbol{\epsilon} \tag{4.3.11}$$

matrix representation of coordinates

$$[\,\sigma_{ij}\,] = \begin{bmatrix} \lambda I_\epsilon + 2\,\mu\epsilon_{11} & 2\,\mu\epsilon_{12} & 2\,\mu\epsilon_{13} \\ 2\,\mu\epsilon_{21} & \lambda I_\epsilon + 2\,\mu\epsilon_{22} & 2\,\mu\epsilon_{23} \\ 2\,\mu\epsilon_{31} & 2\,\mu\epsilon_{32} & \lambda I_\epsilon + 2\,\mu\epsilon_{33} \end{bmatrix} \tag{4.3.12}$$

linear elastic continuum tangent stiffness (constant in strains)

$$\mathbb{E}^{\tan} = \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu \mathbb{I}^{\mathrm{sym}} \qquad \mathrm{D}_t\boldsymbol{\sigma} = \mathbb{E}^{\tan} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.3.13}$$

linear elastic continuum secant stiffness

$$\mathbb{E} = \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu \mathbb{I}^{\mathrm{sym}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.14}$$

Voigt representation of stiffness tensor

$$\mathbb{E} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{4.3.15}$$

volumetric - deviatoric decomposition

specific stored energy (quadratic in volumetric and deviatoric strains)

$$\begin{aligned} W &= W^{\text{vol}}(\boldsymbol{\epsilon}^{\text{vol}}) &+& \; W^{\text{dev}}(\boldsymbol{\epsilon}^{\text{dev}}) \\ &= \tfrac{1}{2}\,\kappa\,[\boldsymbol{\epsilon}^{\text{vol}} : \boldsymbol{I}]^2 &+& \; \mu\,[\boldsymbol{\epsilon}^{\text{dev}2} : \boldsymbol{I}] \end{aligned} \tag{4.3.16}$$

with

$$\begin{aligned} \boldsymbol{\epsilon}^{\text{vol}} &= \tfrac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} &=& \; \mathbb{I}^{\text{vol}} : \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon}^{\text{dev}} &= \boldsymbol{\epsilon} - \tfrac{1}{3}[\boldsymbol{\epsilon} : \boldsymbol{I}]\,\boldsymbol{I} &=& \; \mathbb{I}^{\text{dev}} : \boldsymbol{\epsilon} \end{aligned} \tag{4.3.17}$$

stress tensor

$$\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = \mathbb{E} : \boldsymbol{\epsilon} = 3\,\kappa\,\boldsymbol{\epsilon}^{\text{vol}} + 2\,\mu\,\boldsymbol{\epsilon}^{\text{dev}} \tag{4.3.18}$$

matrix representation of coordinates

$$[\,\sigma_{ij}\,] = \begin{bmatrix} \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{11} & 2\,\mu\epsilon^{\text{dev}}_{12} & 2\,\mu\epsilon^{\text{dev}}_{13} \\ 2\,\mu\epsilon^{\text{dev}}_{21} & \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{22} & 2\,\mu\epsilon^{\text{dev}}_{23} \\ 2\,\mu\epsilon^{\text{dev}}_{31} & 2\,\mu\epsilon^{\text{dev}}_{32} & \kappa I_\epsilon + 2\,\mu\epsilon^{\text{dev}}_{33} \end{bmatrix} \tag{4.3.19}$$

linear elastic continuum tangent stiffness

$$\mathbb{E}^{\text{tan}} = 3\,\kappa\mathbb{I}^{\text{vol}} + 2\,\mu\mathbb{I}^{\text{dev}} \qquad \mathrm{D}_t\boldsymbol{\sigma} = \mathbb{E}^{\text{tan}} : \mathrm{D}_t\boldsymbol{\epsilon} \tag{4.3.20}$$

linear elastic continuum secant stiffness

$$\mathbb{E} = 3\,\kappa \mathbb{I}^{\text{vol}} + 2\,\mu \mathbb{I}^{\text{dev}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.21}$$

Voigt representation of stiffness tensor

$$\mathbb{E} = \begin{bmatrix} \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & \kappa - \frac{2}{3}\mu & 0 & 0 & 0 \\ \kappa - \frac{2}{3}\mu & \kappa - \frac{2}{3}\mu & \kappa + \frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{4.3.22}$$

transformation with fourth order unit tensors

$$\begin{aligned} \mathbb{I}^{\text{vol}} &= \tfrac{1}{3}\,\boldsymbol{I} \otimes \boldsymbol{I} \\ \mathbb{I}^{\text{dev}} &= \mathbb{I}^{\text{sym}} - \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{sym}} - \tfrac{1}{3}\boldsymbol{I} \otimes \boldsymbol{I} \end{aligned} \tag{4.3.23}$$

thus

$$\begin{aligned} \mathbb{E} = \mathbb{E}^{\text{tan}} &= 3\,\kappa\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{dev}} \\ &= 3\,[\,\kappa - \tfrac{2}{3}\mu]\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{sym}} \\ &= 3\,\lambda\,\mathbb{I}^{\text{vol}} + 2\,\mu\,\mathbb{I}^{\text{sym}} \\ &= \lambda \boldsymbol{I} \otimes \boldsymbol{I} + 2\,\mu\,\mathbb{I}^{\text{sym}} \end{aligned} \tag{4.3.24}$$

bulk modulus and shear modulus

$$\kappa = \lambda + \frac{2}{3}\,\mu \qquad \text{and} \qquad \mu \tag{4.3.25}$$

comparison of coordinates E_{1111} and E_{1122}

$$
\begin{aligned}
\kappa + \frac{4}{3}\mu &= \lambda + \frac{2}{3}\mu + \frac{4}{3}\mu = \lambda + 2\,\mu \\
\kappa - \frac{2}{3}\mu &= \lambda + \frac{2}{3}\mu - \frac{2}{3}\mu = \lambda
\end{aligned}
\tag{4.3.26}
$$

restrictions to elastic constants from positive stored energy, i.e. positive definite elastic stiffness

$$
\boldsymbol{\epsilon} : \mathbb{E} : \boldsymbol{\epsilon} > 0 \quad \forall \boldsymbol{\epsilon} \neq \mathbf{0} \quad \rightarrow \quad \kappa, \mu > 0, \lambda > -\frac{2}{3}\mu \tag{4.3.27}
$$

4.3.2 Specific complementary energy

fourth order elasticity tensor

$$\mathbb{E} = 2\,\mu \mathbb{I}^{\text{sym}} + \lambda \boldsymbol{I} \otimes \boldsymbol{I} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.28}$$

inversion by making use of Sherman-Morrison-Woodbury theorem

$$/\!\!\!\mathbb{A} = \mathbb{B} + \alpha \boldsymbol{C} \otimes \boldsymbol{D} \tag{4.3.29}$$

inverse of rank 1 modified fourth order tensor

$$/\!\!\!\mathbb{A}^{-1} = \mathbb{B}^{-1} - \alpha\, \frac{\mathbb{B}^{-1} : \boldsymbol{C} \otimes \boldsymbol{D} : \mathbb{B}^{-1}}{1 + \alpha\, \boldsymbol{D} : \mathbb{B}^{-1} : \boldsymbol{C}} \tag{4.3.30}$$

with $/\!\!\!\mathbb{A} = \mathbb{E}$, $/\!\!\!\mathbb{A}^{-1} = \mathbb{C}$, $\mathbb{B} = 2\,\mu \mathbb{I}^{\text{sym}}$, $\alpha = \lambda$, $\boldsymbol{C} = \boldsymbol{I}$ and $\boldsymbol{D} = \boldsymbol{I}$ we obtain the fourth order compliance tensor

---

$$\mathbb{C} = \frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} - \frac{\lambda}{2\,\mu[2\,\mu + 3\,\lambda]}\boldsymbol{I} \otimes \boldsymbol{I} \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \tag{4.3.31}$$

---

or rather

$$\mathbb{C} = \gamma \boldsymbol{I} \otimes \boldsymbol{I} + \frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} \quad \text{with} \quad \gamma = -\frac{\lambda}{2\mu[2\mu + 3\lambda]} \tag{4.3.32}$$

check
with $\mathbb{I}^{\text{sym}} : \mathbb{I}^{\text{sym}} = \mathbb{I}^{\text{sym}}$, $\mathbb{I}^{\text{sym}} : [\boldsymbol{I} \otimes \boldsymbol{I}] = \boldsymbol{I} \otimes \boldsymbol{I}$ and $[\boldsymbol{I} \otimes \boldsymbol{I}] : [\boldsymbol{I} \otimes \boldsymbol{I}] = 3\,\boldsymbol{I} \otimes \boldsymbol{I}$

$$\begin{aligned} \mathbb{E} : \mathbb{E}^{-1} &= [2\,\mu \mathbb{I}^{\text{sym}} + \lambda \boldsymbol{I} \otimes \boldsymbol{I}] : [\frac{1}{2\,\mu}\mathbb{I}^{\text{sym}} - \frac{\lambda}{2\,\mu[2\,\mu + 3\,\lambda]}\boldsymbol{I} \otimes \boldsymbol{I}] \\ &= \mathbb{I}^{\text{sym}} + [\frac{\lambda}{2\mu} - \frac{2\mu\lambda}{2\mu[2\mu + 3\lambda]} - \frac{3\,\lambda^2}{2\mu[2\mu + 3\lambda]}]\boldsymbol{I} \otimes \boldsymbol{I} = \mathbb{I}^{\text{sym}} \end{aligned} \tag{4.3.33}$$

Voigt representation of compliance tensor

$$\mathbb{C} = \begin{bmatrix} \gamma + \frac{1}{2\mu} & \gamma & \gamma & 0 & 0 & 0 \\ \gamma & \gamma + \frac{1}{2\mu} & \gamma & 0 & 0 & 0 \\ \gamma & \gamma & \gamma + \frac{1}{2\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\mu} \end{bmatrix} \qquad (4.3.34)$$

strain tensor (linear in stresses)

$$\boldsymbol{\epsilon} = \mathrm{D}_{\boldsymbol{\sigma}} W^* = \mathbb{C} : \boldsymbol{\sigma} = \gamma [\, \boldsymbol{\sigma} : \boldsymbol{I} \,] \, \boldsymbol{I} + \frac{1}{2\mu} \boldsymbol{\sigma} \qquad (4.3.35)$$

matrix representation of coordinates

$$[\, \epsilon_{ij} \,] = \begin{bmatrix} \gamma I_\sigma + \frac{1}{2\mu}\sigma_{11} & \frac{1}{2\mu}\sigma_{12} & \frac{1}{2\mu}\sigma_{13} \\ \frac{1}{2\mu}\sigma_{21} & \gamma I_\sigma + \frac{1}{2\mu}\sigma_{22} & \frac{1}{2\mu}\sigma_{23} \\ \frac{1}{2\mu}\sigma_{31} & \frac{1}{2\mu}\sigma_{32} & \gamma I_\sigma + \frac{1}{2\mu}\sigma_{33} \end{bmatrix} \qquad (4.3.36)$$

specific complementary energy (quadratic in stresses)

$$W^* = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C} : \boldsymbol{\sigma} = \frac{1}{2} \gamma \, [\boldsymbol{\sigma} : \boldsymbol{I}]^2 + \frac{1}{4\mu} [\boldsymbol{\sigma}^2 : \boldsymbol{I}] \qquad (4.3.37)$$

volumetric - deviatoric decomposition

fourth order elasticity tensor

$$\mathbb{E} = 2\,\mu \mathbb{I}^{\text{dev}} + 3\,\kappa \mathbb{I}^{\text{vol}} \qquad \boldsymbol{\sigma} = \mathbb{E} : \boldsymbol{\epsilon} \tag{4.3.38}$$

inversion by making use of orthogonality of $\mathbb{I}^{\text{vol}}$ and $\mathbb{I}^{\text{dev}}$ yields fourth order compliance tensor

---

$$\mathbb{C} = \frac{1}{2\,\mu} \mathbb{I}^{\text{dev}} + \frac{1}{3\,\kappa} \mathbb{I}^{\text{vol}} \qquad \boldsymbol{\epsilon} = \mathbb{C} : \boldsymbol{\sigma} \tag{4.3.39}$$

---

check
with $\mathbb{I}^{\text{dev}} : \mathbb{I}^{\text{dev}} = \mathbb{I}^{\text{dev}}$, $\mathbb{I}^{\text{vol}} : \mathbb{I}^{\text{vol}} = \mathbb{I}^{\text{vol}}$ and $\mathbb{I}^{\text{dev}} : \mathbb{I}^{\text{vol}} = \mathbb{O}$

$$\begin{aligned} \mathbb{E} : \mathbb{E}^{-1} &= \left[2\,\mu \mathbb{I}^{\text{dev}} + 3\,\kappa\, \mathbb{I}^{\text{vol}}\right] : \left[\frac{1}{2\,\mu} \mathbb{I}^{\text{dev}} + \frac{1}{3\,\kappa} \mathbb{I}^{\text{vol}}\right] \\ &= \mathbb{I}^{\text{dev}} + \mathbb{I}^{\text{vol}} = \mathbb{I} \end{aligned} \tag{4.3.40}$$

Voigt representation of compliance

$$\mathbb{C} = \begin{bmatrix} \frac{1}{9\kappa} + \frac{2}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & 0 & 0 & 0 \\ \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} + \frac{2}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & 0 & 0 & 0 \\ \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} - \frac{1}{6\mu} & \frac{1}{9\kappa} + \frac{2}{6\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4\mu} \end{bmatrix} \tag{4.3.41}$$

comparison of coordinates C_{1111} and C_{1122}

$$\begin{aligned}
\frac{1}{9\kappa}+\frac{2}{6\mu} &= \frac{1}{9\lambda+6\mu}+\frac{2}{6\mu}=\frac{1}{3[3\lambda+2\mu]}-\frac{1}{3[2\mu]}+\frac{3}{3[2\mu]}\\
&= \frac{2\mu-3\lambda-2\mu}{3[2\mu][3\lambda+2\mu]}+\frac{1}{2\mu}\\
&= \frac{-\lambda}{2\mu[2\mu+3\lambda]}+\frac{1}{2\mu}=\gamma+\frac{1}{2\mu}\\
\frac{1}{9\kappa}-\frac{1}{6\mu} &= \frac{1}{9\lambda+6\mu}-\frac{1}{6\mu}=\frac{1}{3[3\lambda+2\mu]}-\frac{1}{3[2\mu]}\\
&= \frac{2\mu-3\lambda-2\mu}{3[2\mu][3\lambda+2\mu]}=-\frac{\lambda}{2\mu[2\mu+3\lambda]}=\gamma
\end{aligned} \tag{4.3.42}$$

strain tensor (linear in stresses)

$$\boldsymbol{\epsilon}=\mathrm{D}_{\boldsymbol{\sigma}}W^{*}=\mathbb{C}:\boldsymbol{\sigma}=\kappa[\,\boldsymbol{\sigma}:\boldsymbol{I}\,]\,\boldsymbol{I}+\frac{1}{4\mu}\,\boldsymbol{\sigma}^{\mathrm{dev}} \tag{4.3.43}$$

matrix representation of coordinates

$$[\,\epsilon_{ij}\,]=\begin{bmatrix}\kappa I_{\epsilon}+2\,\mu\epsilon_{11}^{\mathrm{dev}} & 2\,\mu\epsilon_{12}^{\mathrm{dev}} & 2\,\mu\epsilon_{13}^{\mathrm{dev}}\\ 2\,\mu\epsilon_{21}^{\mathrm{dev}} & \kappa I_{\epsilon}+2\,\mu\epsilon_{22}^{\mathrm{dev}} & 2\,\mu\epsilon_{23}^{\mathrm{dev}}\\ 2\,\mu\epsilon_{31}^{\mathrm{dev}} & 2\,\mu\epsilon_{32}^{\mathrm{dev}} & \kappa I_{\epsilon}+2\,\mu\epsilon_{33}^{\mathrm{dev}}\end{bmatrix} \tag{4.3.44}$$

specific stored energy (quadratic in volumetric and deviatoric stresses)

$$\begin{aligned}
W^{*} &= W^{*\mathrm{vol}}(\boldsymbol{\sigma}^{\mathrm{vol}}) + W^{*\mathrm{dev}}(\boldsymbol{\sigma}^{\mathrm{dev}})\\
&= \tfrac{1}{2}\,\kappa\,\epsilon^{\mathrm{vol}2} + \mu\,[\boldsymbol{\epsilon}^{\mathrm{dev}2}:\boldsymbol{I}]
\end{aligned} \tag{4.3.45}$$

4.3.3 Elastic constants

isotropic linear elasticity can be characterized by only two elastic constants

	E,ν	E,μ	λ,μ	κ,μ
E	E	E	$\dfrac{\mu\,[3\lambda+2\mu]}{\lambda+\mu}$	$\dfrac{9\kappa\mu}{3\kappa+\mu}$
ν	ν	$\dfrac{E-2\mu}{2\mu}$	$\dfrac{\lambda}{2[\lambda+\mu]}$	$\dfrac{3\kappa-2\mu}{6\kappa+2\mu}$
μ	$\dfrac{E}{2[1+\nu]}$	μ	μ	μ
λ	$\dfrac{E\nu}{[1+\nu][1-2\nu]}$	$\dfrac{\mu[E-2\mu]}{3\mu-E}$	λ	$\kappa-\dfrac{2}{3}\mu$
κ	$\dfrac{E}{3[1-2\nu]}$	$\dfrac{E\mu}{3[3\mu-E]}$	$\lambda+\dfrac{2}{3}\mu$	κ

Table 4.1: relations betweeb elastic constants

4.4 Transversely isotropic hyperelasticity

fiber direction $\boldsymbol{n}$ and structural tensor $\boldsymbol{N}$

$$\boldsymbol{N} = \boldsymbol{n} \otimes \boldsymbol{n} \qquad \text{with} \qquad |\boldsymbol{n}| = 1 \tag{4.4.1}$$

specific stored energy:
isotropic tensor function with two arguments

$$\begin{aligned} W &= W(\boldsymbol{\epsilon}, \boldsymbol{n}) = W(\boldsymbol{\epsilon}, -\boldsymbol{n}) = W(\boldsymbol{\epsilon}, \boldsymbol{N}) \\ &= W(\boldsymbol{Q} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{Q}^{\mathrm{t}}, \boldsymbol{Q} \cdot \boldsymbol{N} \cdot \boldsymbol{Q}^{\mathrm{t}}) \qquad \forall\, \boldsymbol{Q} \in SO(3) \end{aligned} \tag{4.4.2}$$

representation theorem for isotropic tensor functions with two arguments

$$W = W(\boldsymbol{\epsilon}, \boldsymbol{N}) = W(i_\epsilon, i_N, i_{\epsilon N},) \tag{4.4.3}$$

irreducible set of ten invariants, integrity basis

$$\begin{aligned} \bar{i}_\epsilon &= \{\boldsymbol{\epsilon} : \boldsymbol{I}, \boldsymbol{\epsilon}^2 : \boldsymbol{I}, \boldsymbol{\epsilon}^3 : \boldsymbol{I}\} \\ i_N &= \{\boldsymbol{N} : \boldsymbol{I}, \boldsymbol{N}^2 : \boldsymbol{I}, \boldsymbol{N}^3 : \boldsymbol{I}\} \\ i_{\epsilon,N} &= \{\boldsymbol{\epsilon} : \boldsymbol{N}, \boldsymbol{\epsilon} : \boldsymbol{N}^2, \boldsymbol{\epsilon}^2 : \boldsymbol{N}, \boldsymbol{\epsilon}^2 : \boldsymbol{N}^2\} \end{aligned} \tag{4.4.4}$$

recall different representation of set of three invariants

$$\begin{aligned} \bar{i}_\epsilon &= \{\boldsymbol{\epsilon} : \boldsymbol{I}, \boldsymbol{\epsilon}^2 : \boldsymbol{I}, \boldsymbol{\epsilon}^3 : \boldsymbol{I}\} && \text{basic invariants} \\ i_\epsilon &= \{\mathrm{tr}(\boldsymbol{\epsilon}), \tfrac{1}{2}[\mathrm{tr}^2(\boldsymbol{\epsilon}) - \mathrm{tr}(\boldsymbol{\epsilon}^2)], \det(\boldsymbol{\epsilon})\} && \text{principal invariants} \\ i_\epsilon &= \{(\lambda_{\epsilon 1} + \lambda_{\epsilon 2} + \lambda_{\epsilon 3}), (\lambda_{\epsilon 2}\lambda_{\epsilon 3} + \lambda_{\epsilon 3}\lambda_{\epsilon 1} + \lambda_{\epsilon 1}\lambda_{\epsilon 3}), (\lambda_{\epsilon 1}\lambda_{\epsilon 2}\lambda_{\epsilon 3})\} \\ & && \text{eigenvalue representation of principal invariants} \end{aligned} \tag{4.4.5}$$

until now: principal invariants i_ϵ, now: basic invariants $\bar{i}_\epsilon$

with properties of structural tensor, idempotence & normalization

$$\begin{aligned} \boldsymbol{N}^2 &= \boldsymbol{N}\cdot\boldsymbol{N} = [\boldsymbol{n}\otimes\boldsymbol{n}]\cdot[\boldsymbol{n}\otimes\boldsymbol{n}] = \boldsymbol{n}\otimes\boldsymbol{n} = \boldsymbol{N} \\ \boldsymbol{N}:\boldsymbol{I} &= \boldsymbol{n}\cdot\boldsymbol{n} = |\boldsymbol{n}|^2 = 1 \end{aligned} \tag{4.4.6}$$

reduced representation with five invariants

$$W = W(\boldsymbol{\epsilon},\boldsymbol{N}) = W(\bar{I}_\epsilon, \bar{II}_\epsilon, \bar{III}_\epsilon, \bar{IV}_\epsilon, \bar{V}_\epsilon) \tag{4.4.7}$$

representation theorem for isotropic tensor functions with two arguments

$$\boldsymbol{\sigma}(\boldsymbol{\epsilon},\boldsymbol{N}) = f_1\boldsymbol{I} + f_2\boldsymbol{\epsilon} + f_3\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} + f_4\boldsymbol{N} + f_5[\boldsymbol{\epsilon}\cdot\boldsymbol{N} + \boldsymbol{N}\cdot\boldsymbol{\epsilon}] \tag{4.4.8}$$

stress for transversaly isotropic hyperelastic material

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}}W &= \frac{\mathrm{D}W}{\mathrm{D}\bar{I}_\epsilon}\frac{\mathrm{D}\bar{I}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{II}_\epsilon}\frac{\mathrm{D}\bar{II}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{III}_\epsilon}\frac{\mathrm{D}\bar{III}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \\ &= \frac{\mathrm{D}W}{\mathrm{D}\bar{IV}_\epsilon}\frac{\mathrm{D}\bar{IV}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} + \frac{\mathrm{D}W}{\mathrm{D}\bar{V}_\epsilon}\frac{\mathrm{D}\bar{V}_\epsilon}{\mathrm{D}\boldsymbol{\epsilon}} \end{aligned} \tag{4.4.9}$$

with derivatives of invariants $\bar{I}_\epsilon$, $\bar{II}_\epsilon$, $\bar{III}_\epsilon$, $\bar{IV}_\epsilon$,$\bar{V}_\epsilon$ with respect to second order tensor $\boldsymbol{\epsilon}$

$$\begin{array}{rclll} \bar{I}_\epsilon &=& \boldsymbol{\epsilon} : \boldsymbol{I} & \text{linear} & \mathrm{D}_{\boldsymbol{\epsilon}}\bar{I}_\epsilon = \boldsymbol{I} \\ \bar{II}_\epsilon &=& \boldsymbol{\epsilon}^2 : \boldsymbol{I} & \text{quadr.} & \mathrm{D}_{\boldsymbol{\epsilon}}\bar{II}_\epsilon = 2\,\boldsymbol{\epsilon} \\ \bar{III}_\epsilon &=& \boldsymbol{\epsilon}^3 : \boldsymbol{I} & \text{cubic} & \mathrm{D}_{\boldsymbol{\epsilon}}\bar{III}_\epsilon = 3\,\boldsymbol{\epsilon}^2 \\ \bar{IV}_\epsilon &=& \boldsymbol{\epsilon} : \boldsymbol{N} & \text{linear} & \mathrm{D}_{\boldsymbol{\epsilon}}\bar{IV}_\epsilon = \boldsymbol{N} \\ \bar{V}_\epsilon &=& \boldsymbol{\epsilon}^2 : \boldsymbol{N} & \text{quadr.} & \mathrm{D}_{\boldsymbol{\epsilon}}\bar{V}_\epsilon = \boldsymbol{\epsilon}\cdot\boldsymbol{N} + \boldsymbol{N}\cdot\boldsymbol{\epsilon} \end{array} \tag{4.4.10}$$

with $\bar{V}_\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N} = [\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}] : \boldsymbol{N} = \boldsymbol{\epsilon} : [\boldsymbol{\epsilon}\cdot\boldsymbol{N}] = \boldsymbol{\epsilon} : [\boldsymbol{N}\cdot\boldsymbol{\epsilon}]$

alternatively
$I\bar{V}_\epsilon = \boldsymbol{\epsilon} : \boldsymbol{N} = \boldsymbol{n} \cdot \boldsymbol{\epsilon} \cdot \boldsymbol{n}$ and $\bar{V}_\epsilon = \boldsymbol{\epsilon}^2 : \boldsymbol{N} = \boldsymbol{n} \cdot \boldsymbol{\epsilon}^2 \cdot \boldsymbol{n}$
general represenation of stress

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W &= \mathrm{D}_{\bar{I}_\epsilon} W \, \boldsymbol{I} + 2\,\mathrm{D}_{\bar{II}_\epsilon} W \, \boldsymbol{\epsilon} + 3\,\mathrm{D}_{I\bar{II}_\epsilon} W \, \boldsymbol{\epsilon}^2 \\ &= \mathrm{D}_{I\bar{V}_\epsilon} W \, \boldsymbol{N} + \mathrm{D}_{\bar{V}_\epsilon} W \, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}] \end{aligned} \tag{4.4.11}$$

comparison of coefficients

assumption of linearity (quadratic term vanishes), six material parameters $f_{11}, f_{14}, f_2, f_{41}, f_{44}, f_5$

$$\begin{aligned} f_1 &= \ \mathrm{D}_{\bar{I}_\epsilon} W &&= f_{11} \bar{I}_\epsilon + f_{14} I\bar{V}_\epsilon \\ f_2 &= 2\ \mathrm{D}_{\bar{II}_\epsilon} W &&= \text{const.} \\ f_3 &= 3\ \mathrm{D}_{I\bar{II}_\epsilon} W &&= 0 \\ f_4 &= \ \mathrm{D}_{I\bar{V}_\epsilon} W &&= f_{41} \bar{I}_\epsilon + f_{44} I\bar{V}_\epsilon \\ f_5 &= \ \mathrm{D}_{\bar{V}_\epsilon} W &&= \text{const.} \end{aligned} \tag{4.4.12}$$

specific stored energy (quadratic in strains)

$$\begin{aligned} W = \frac{1}{2} \boldsymbol{\sigma} : \boldsymbol{\epsilon} &= \frac{1}{2} f_{11} \, \bar{I}_\epsilon^2 + \frac{1}{2} f_{14} \, I\bar{V}_\epsilon \bar{I}_\epsilon + \frac{1}{2} f_2 \, \bar{II}_\epsilon \\ &+ \frac{1}{2} f_{41} \, \bar{I}_\epsilon I\bar{V}_\epsilon + \frac{1}{2} f_{44} \, I\bar{V}_\epsilon^2 + f_5 \, \bar{V}_\epsilon^2 \end{aligned} \tag{4.4.13}$$

stress tensor (linear in strains)

$$\begin{aligned} \boldsymbol{\sigma} = \mathrm{D}_\epsilon W = [f_{11} \bar{I}_\epsilon + f_{14} I\bar{V}_\epsilon] \, \boldsymbol{I} &+ f_2 \, \boldsymbol{\epsilon} \\ + [f_{41} \bar{I}_\epsilon + f_{44} I\bar{V}_\epsilon] \, \boldsymbol{N} &+ f_5 \, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}] \end{aligned} \tag{4.4.14}$$

linear elastic continuum tangent stiffness (constant in strains)
$\mathbb{E}^{\tan} = \mathrm{D}_{\boldsymbol{\epsilon}} \boldsymbol{\sigma}$

$$\mathbb{E}^{\tan} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} \boldsymbol{I} \otimes \boldsymbol{N} + f_{41} \boldsymbol{N} \otimes \boldsymbol{I} + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\mathrm{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.15}$$

whereby $\not{\mathbb{A}} = 2\,\mathrm{D}_{\boldsymbol{\epsilon}}[\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}]$

due to symmetry reduction to five material parameters $f_{14} = f_{41}$

$$\mathbb{E}^{\tan} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} [\boldsymbol{I} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{I}] + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\mathrm{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.16}$$

---

linear elastic continuum secant stiffness

$$\mathbb{E} = f_{11} \boldsymbol{I} \otimes \boldsymbol{I} + f_{14} [\boldsymbol{I} \otimes \boldsymbol{N} + \boldsymbol{N} \otimes \boldsymbol{I}] + f_{44} \boldsymbol{N} \otimes \boldsymbol{N} + f_2 \mathbb{I}^{\mathrm{sym}} + f_5 \not{\mathbb{A}} \tag{4.4.17}$$

Physical interpretation of parameters

interpretation of f_{11}, $f_{14} = f_{41}$, f_{44}, f_2 and f_5 Spencer [1984]

$$W = \frac{1}{2} \lambda \, \bar{I}_{\epsilon}^2 + \alpha \, \bar{I}_{\epsilon} I\bar{V}_{\epsilon} + \frac{1}{2} \beta \, I\bar{V}_{\epsilon}^2 + \mu_{\perp} \, \bar{I}I_{\epsilon} + 2[\mu_{\parallel} - \mu_{\perp}] \bar{V}_{\epsilon}^2 \tag{4.4.18}$$

---

stress tensor (linear in strains)

$$\begin{aligned}\boldsymbol{\sigma} = \mathrm{D}_{\boldsymbol{\epsilon}} W = [\lambda\, \bar{I}_\epsilon + \alpha\, I\bar{V}_\epsilon]\, \boldsymbol{I} \quad + 2\, \mu_\perp\, \boldsymbol{\epsilon} \\ + [\alpha\, \bar{I}_\epsilon + \beta\, I\bar{V}_\epsilon]\, \boldsymbol{N} + 2\, [\mu_\parallel - \mu_\perp]\, [\boldsymbol{\epsilon} \cdot \boldsymbol{N} + \boldsymbol{N} \cdot \boldsymbol{\epsilon}]\end{aligned} \tag{4.4.19}$$

Voigt representation of stiffness tensor for $\boldsymbol{n} = [\,1, 0, 0\,]^{\mathrm{t}}$, i.e. transverse isotropy with respect to the $\boldsymbol{e}_1$ axis

$$\mathbb{E} = \begin{bmatrix} \begin{matrix}\lambda + 2\alpha + 4\mu_\parallel \\ -2\mu_\perp + \beta\end{matrix} & \lambda + \alpha & \lambda + \alpha & 0 & 0 & 0 \\ \lambda + \alpha & \lambda + 2\mu_\perp & \lambda & 0 & 0 & 0 \\ \lambda + \alpha & \lambda & \lambda + 2\mu_\perp & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_\parallel & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_\perp & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_\parallel \end{bmatrix} \tag{4.4.20}$$

comparison of coefficients

$$f_{11} = \lambda \quad f_2 = 2\, \mu_\perp \quad f_{14} = \alpha = f_{41} \quad f_{44} = \beta \quad f_5 = 2\, [\mu_\parallel - \mu_\perp] \tag{4.4.21}$$

with shear moduli $\mu_\parallel$ for shear in fiber direction and $\mu_\perp$ for shear parallel to the fiber direction

A Übungsaufgaben

A.1 Tensoralgebra zweistufiger Tensoren

Aufgabe: Bestimmen Sie die Wurzel des symmetrischen Anteils des zweistufigen Tensors $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad [A]_{ij} = \begin{bmatrix} 10 & 16 & -10 \\ -4 & 10 & 8 \\ 10 & -8 & 1 \end{bmatrix}$$

---

symmetrisch / schiefsymmetrische Zerlegung von $\boldsymbol{A}$

$$\boldsymbol{S} = \frac{1}{2}\left[\boldsymbol{A} + \boldsymbol{A}^{\mathrm{t}}\right] \qquad \boldsymbol{W} = \frac{1}{2}\left[\boldsymbol{A} - \boldsymbol{A}^{\mathrm{t}}\right]$$

mit symmetrischem Anteil $\boldsymbol{S}$

$$\boldsymbol{S} = S_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad [S]_{ij} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

und antimetrischem Anteil $\boldsymbol{W}$

$$\boldsymbol{W} = W_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad [W]_{ij} = \begin{bmatrix} 0 & 10 & -10 \\ -10 & 0 & 8 \\ 10 & -8 & 0 \end{bmatrix}$$

Quadrat des symmetrischen Tensors $\boldsymbol{S}$

$$\begin{aligned}
\boldsymbol{S}^2 &= \left(S_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right) \cdot \left(S_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l\right) \\
&= S_{ij}\, S_{kl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \cdot \boldsymbol{e}_k \otimes \boldsymbol{e}_l \\
&= S_{ij}\, S_{kl}\delta_{jk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l \\
&= S_{ij}\, S_{jl}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l = [S^2]_{il}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l
\end{aligned}$$

$$[S]_{jl} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[S]_{ij} = \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 136 & 120 & 0 \\ 120 & 136 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [S^2]_{il}$$

Bestimmung der Invarianten des zweitstufigen Tensors S

$$\begin{aligned}
I_S &= \mathrm{tr}(\boldsymbol{S}) = \boldsymbol{S} : \boldsymbol{I} \\
&= S_{11} + S_{22} + S_{33} \\
&= 10 + 10 + 1 = \underline{\underline{21}} \\
II_S &= \tfrac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{S}) - \mathrm{tr}(\boldsymbol{S}^2)\right] = \tfrac{1}{2}\left[[\boldsymbol{S} : \boldsymbol{I}]^2 - [\boldsymbol{S}^2 : \boldsymbol{I}]\right] \\
&= \tfrac{1}{2}\left[I_S^2 - [S^2]_{11} - [S^2]_{22} - [S^2]_{33}\right] \\
&= \tfrac{1}{2}[441 - 136 - 136 - 1] = \underline{\underline{84}} \\
III_S &= \det(\boldsymbol{S}) \\
&= S_{11}S_{22}S_{33} + S_{21}S_{32}S_{13} + S_{31}S_{12}S_{23} \\
&- S_{11}S_{23}S_{32} - S_{22}S_{31}S_{13} - S_{33}S_{12}S_{21} \\
&= 100 + 0 + 0 - 0 - 0 - 36 = \underline{\underline{64}}
\end{aligned}$$

Charakteristisches Polynom

Eigenwerte / Nullstellen des charakteristischen Polynoms

$$\lambda^3 - I_S\,\lambda^2 + II_S\,\lambda - III_S = 0$$

mit Invarianten $I_S = 21$, $II_S = 84$, $III_S = 64$

$$\lambda^3 - 21\,\lambda^2 + 84\,\lambda - 64 = 0$$

Eigenwerte des zweitstufigen Tensors S

kubische Gleichung, 'testen' des ersten Eigenwertes $\lambda = 1$

$$
\begin{array}{l}
(\lambda^3 - 21\,\lambda^2 + 84\,\lambda - 64\,) : (\lambda - 1) = \underline{\lambda^2 - 20\,\lambda + 64} \\
\lambda^3 - \quad \lambda^2 \\
\quad - 20\,\lambda^2 + 84\,\lambda \\
\quad - 20\,\lambda^2 + 20\,\lambda \\
\qquad\qquad + 64\,\lambda - 64 \\
\qquad\qquad + 64\,\lambda - 64
\end{array}
$$

quadratische Restgleichung, zweiter und dritter Eigenwert

$$\lambda^2 - 20\,\lambda + 64 = (\lambda - 4)(\lambda - 16)$$

Eigenwertdarstellung des charakteristischen Polynoms

$$\lambda^3 - 21\,\lambda^2 + 84\,\lambda - 64 = (\lambda - 1)(\lambda - 4)(\lambda - 16) = 0$$

Eigenwerte des zweitstufigen Tensors $\boldsymbol{S}$

$$\underline{\underline{\lambda_{S1} = 1}} \qquad \underline{\underline{\lambda_{S2} = 4}} \qquad \underline{\underline{\lambda_{S3} = 16}}$$

Eigenvektoren des zweitstufigen Tensors S

- Eigenwert $\boldsymbol{n}_{S1}$ zum Eigenvektor $\lambda_{S1} = 1$

$$[\quad \boldsymbol{S} - \lambda_{S1}\,\boldsymbol{I} \quad] \cdot \boldsymbol{n}_{S1} = \boldsymbol{0}$$

$$\begin{bmatrix} 9 & 6 & 0 \\ 6 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Wahl einer Komponente beliebig, z.B. $n_z = 1$, damit Gleichung (3) erfüllt, aus Gleichung (1) und (2) nur 'triviale' Lösung $n_x = 0$ und $n_y = 0$, abschliessendes Normieren mit $||[0,0,1]|| = 1$ (entfällt), also

$$\lambda_{S1} = 1 \qquad \underline{\underline{\boldsymbol{n}_{S1} = [\,0,0,+1\,]^{\mathrm{t}}}}$$

- Eigenwert $\boldsymbol{n}_{S2}$ zum Eigenvektor $\lambda_{S2} = 4$

$$\left[\quad \boldsymbol{S} - \lambda_{S2}\,\boldsymbol{I} \quad\right] \cdot \boldsymbol{n}_{S2} = \boldsymbol{0}$$

$$\begin{bmatrix} 6 & 6 & 0 \\ 6 & 6 & 0 \\ 0 & 0 & -3 \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Wahl einer Komponente beliebig, z.B. $n_x = 1$, aus Gleichung (1) und (2) $n_y = -1$, aus Gleichung (3) $n_z = 0$, abschliessendes Normieren mit $||[1,-1,0]|| = \sqrt{2}$, also

$$\lambda_{S2} = 4 \qquad \underline{\underline{\boldsymbol{n}_{S2} = \frac{1}{\sqrt{2}}[\,+1,-1,0\,]^{\mathrm{t}}}}$$

- Eigenwert $\boldsymbol{n}_{S3}$ zum Eigenvektor $\lambda_{S3} = 16$

$$\left[\quad \boldsymbol{S} - \lambda_{S3}\,\boldsymbol{I} \quad\right] \cdot \boldsymbol{n}_{S3} = \boldsymbol{0}$$

$$\begin{bmatrix} -6 & -10 & 0 \\ -10 & -6 & 0 \\ 0 & 0 & -15 \end{bmatrix} \cdot \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Wahl einer Komponente beliebig, z.B. $n_x = 1$, aus Gleichung (1) und (2) $n_y = 1$, aus Gleichung (3) $n_z = 0$, abschliessendes Normieren mit $||[1,1,0]|| = \sqrt{2}$, also

$$\lambda_{S3} = 16 \qquad \underline{\underline{\boldsymbol{n}_{S3} = \frac{1}{\sqrt{2}}[\,+1,+1,0\,]^{\mathrm{t}}}}$$

Spektraldarstellung des zweitstufigen Tensors S

Spektraldarstellung des zweitstufigen Tensors $\boldsymbol{S}$

$$
\begin{aligned}
\boldsymbol{S} &= \textstyle\sum_{i=1}^{3} \lambda_{Ai}\, \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai} \\
&= \lambda_{S1}\, \boldsymbol{n}_{S1} \otimes \boldsymbol{n}_{S1} + \lambda_{S2}\, \boldsymbol{n}_{S2} \otimes \boldsymbol{n}_{S2} + \lambda_{S3}\, \boldsymbol{n}_{S3} \otimes \boldsymbol{n}_{S3} \\
&= 1\, \boldsymbol{n}_{S1} \otimes \boldsymbol{n}_{S1} + 4\, \boldsymbol{n}_{S2} \otimes \boldsymbol{n}_{S2} + 16\, \boldsymbol{n}_{S3} \otimes \boldsymbol{n}_{S3}
\end{aligned}
$$

$$
\begin{array}{cccc}
 & 1 & \sqrt{\lambda} & \lambda \\
\boldsymbol{n}_1 \otimes \boldsymbol{n}_1 = & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{bmatrix} \\
\boldsymbol{n}_2 \otimes \boldsymbol{n}_2 = & \begin{bmatrix} +\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} +1 & -1 & 0 \\ -1 & +1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} +2 & -2 & 0 \\ -2 & +2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
\boldsymbol{n}_3 \otimes \boldsymbol{n}_3 = & \begin{bmatrix} +\frac{1}{2} & +\frac{1}{2} & 0 \\ +\frac{1}{2} & +\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} +2 & +2 & 0 \\ +2 & +2 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} +8 & +8 & 0 \\ +8 & +8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 & & \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 & & [\sqrt{S}]_{ij} & [S]_{ij}
\end{array}
$$

Wurzel des zweitstufigen Tensors $\boldsymbol{S}$

$$\begin{aligned}\sqrt{\boldsymbol{S}} &= \textstyle\sum_{i=1}^{3} \sqrt{\lambda_{Ai}}\, \boldsymbol{n}_{Ai} \otimes \boldsymbol{n}_{Ai} \\ &= \sqrt{\lambda_{S1}}\, \boldsymbol{n}_{S1} \otimes \boldsymbol{n}_{S1} + \sqrt{\lambda_{S2}}\, \boldsymbol{n}_{S2} \otimes \boldsymbol{n}_{S2} + \sqrt{\lambda_{S3}}\, \boldsymbol{n}_{S3} \otimes \boldsymbol{n}_{S3} \\ &= 1\, \boldsymbol{n}_{S1} \otimes \boldsymbol{n}_{S1} + 2\, \boldsymbol{n}_{S2} \otimes \boldsymbol{n}_{S2} + 4\, \boldsymbol{n}_{S3} \otimes \boldsymbol{n}_{S3}\end{aligned}$$

$$\sqrt{\boldsymbol{S}} = \sqrt{S}_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \qquad [\sqrt{S}]_{ij} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Kontrolle

$$\sqrt{\boldsymbol{S}}^2 = (\sqrt{S}_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (\sqrt{S}_{kl} \boldsymbol{e}_k \otimes \boldsymbol{e}_l) = [S]_{il}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_l$$

$$[\sqrt{S}]_{jl} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\sqrt{S}]_{ij} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 10 & 6 & 0 \\ 6 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [S]_{il}$$

A.2 Tensoranalysis

Aufgabe: Bestimmen Sie mit Hilfe der Gateaux Ableitung

$$\mathrm{D}\Phi(\boldsymbol{A}) : \Delta\boldsymbol{A} = \frac{\partial\Phi(\boldsymbol{A})}{\partial\boldsymbol{A}} : \Delta\boldsymbol{A} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0}$$

die partiellen Ableitungen der Invarianten I_A, II_A, III_A eines zweistufigen Tensors bezüglich des Tensors selbst

Ableitung der ersten Invariante

$$
\begin{aligned}
I_A &= \mathrm{tr}\,(\boldsymbol{A}) \\
\mathrm{D}I_A : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}|_{\epsilon=0} \\
&= \quad \Delta\boldsymbol{A} : \boldsymbol{I}|_{\epsilon=0} = \boldsymbol{I} : \Delta\boldsymbol{A} \\
\frac{\partial I_A(\boldsymbol{A})}{\partial\boldsymbol{A}} &= \underline{\underline{\boldsymbol{I}}}
\end{aligned}
$$

Ableitung der zweiten Invariante

$$
\begin{aligned}
II_A &= \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{A}) + \mathrm{tr}(\boldsymbol{A}^2)\right] \\
\mathrm{D}II_A : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}\mathrm{tr}^2(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}) - \tfrac{1}{2}\mathrm{tr}(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})^2|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\,\tfrac{1}{2}[[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]^2 - \tfrac{1}{2}[\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : [\boldsymbol{A}^{\mathrm{t}} + \epsilon\Delta\boldsymbol{A}^{\mathrm{t}}]|_{\epsilon=0} \\
&= \quad [\boldsymbol{A} + \epsilon\Delta\boldsymbol{A}] : \boldsymbol{I}]\Delta\boldsymbol{A} : \boldsymbol{I} \\
&- \quad \tfrac{1}{2}\Delta\boldsymbol{A} : \boldsymbol{A}^{\mathrm{t}} - \tfrac{1}{2}\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}} - \epsilon\Delta\boldsymbol{A} : \Delta\boldsymbol{A}^{\mathrm{t}}|_{\epsilon=0} \\
&= \quad [\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}] : \Delta\boldsymbol{A}
\end{aligned}
$$

$$\frac{\partial II_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\mathrm{tr}(\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^{\mathrm{t}}}}$$

mit

$$\mathrm{tr}(\boldsymbol{A}^2) = (\boldsymbol{A} \cdot \boldsymbol{A}) : \boldsymbol{I} =$$

Ableitung der dritten Invariante

$$III_A = \det(\boldsymbol{A})$$

$$\begin{aligned}
\mathrm{D}III_A : \Delta\boldsymbol{A} &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A} + \epsilon\Delta\boldsymbol{A})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A} \cdot [\boldsymbol{I} + \boldsymbol{A}^{-1} \cdot \epsilon\Delta\boldsymbol{A}])|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A}) \cdot \det(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A} + \boldsymbol{I})|_{\epsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\det(\boldsymbol{A}) \\
&\cdot \quad [(\epsilon\lambda_{A^{-1}\cdot\Delta A1} + 1)(\epsilon\lambda_{A^{-1}\cdot\Delta A2} + 1)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} + 1)]\,|_{\epsilon=0} \\
&= \det(\boldsymbol{A}) \cdot [\lambda_{A^{-1}\cdot\Delta A1} + \lambda_{A^{-1}\cdot\Delta A2} + \lambda_{A^{-1}\cdot\Delta A3}] \\
&= \det(\boldsymbol{A}) \cdot \mathrm{tr}(\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A}) \\
&= \det(\boldsymbol{A}) \cdot \boldsymbol{A}^{-\mathrm{t}} : \Delta\boldsymbol{A}
\end{aligned}$$

$$\frac{\partial III_A(\boldsymbol{A})}{\partial \boldsymbol{A}} = \underline{\underline{\det(\boldsymbol{A}) \cdot \boldsymbol{A}^{-\mathrm{t}}}}$$

mit Darstellung der Determinante des Tensors $(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A})$ als charakteristisches Polynom für den Eigenwert $\lambda = -1$

$$\begin{aligned}
\det(\epsilon\boldsymbol{A}^{-1} \cdot \Delta\boldsymbol{A} - \lambda\boldsymbol{I}) \;&=\; (\epsilon\lambda_{A^{-1}\cdot\Delta A1} - \lambda) \\
&\quad\; (\epsilon\lambda_{A^{-1}\cdot\Delta A2} - \lambda)(\epsilon\lambda_{A^{-1}\cdot\Delta A3} - \lambda)
\end{aligned}$$

Umformung, dazu Übergang auf Indexnotation

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A}) &= (\boldsymbol{A}^{-1}\cdot\Delta\boldsymbol{A}):\boldsymbol{I}\\
&= (A^{-1}_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\cdot(\Delta A_{kl}\boldsymbol{e}_k\otimes\boldsymbol{e}_l):(\delta_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\
&= (A^{-1}_{ij}\Delta A_{jl}\boldsymbol{e}_i\otimes\boldsymbol{e}_l):(\delta_{mn}\boldsymbol{e}_m\otimes\boldsymbol{e}_n)\\
&= A^{-1}_{ij}\Delta A_{jl}\delta_{im}\delta_{ln}\delta_{mn} = \underline{\underline{\underline{\boldsymbol{A}^{-\mathrm{t}}:\Delta\boldsymbol{A}}}}
\end{aligned}$$

A.3 Ableitungen

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zu zeigen: die Ableitung der Funktion $\Phi(\boldsymbol{A})$ nach ihrem tensoriellen Argument $\boldsymbol{A}$ erhält man durch komponentenweise Ableiten nach den einzelnen Tensoreinträgen

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Indexdarstellung des Tensors $\boldsymbol{A}$

$$\boldsymbol{A} = A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

$$\begin{aligned}
\boldsymbol{A} &= A_{11}\boldsymbol{e}_1\otimes\boldsymbol{e}_1 + A_{12}\boldsymbol{e}_1\otimes\boldsymbol{e}_2 + A_{13}\boldsymbol{e}_1\otimes\boldsymbol{e}_3\\
&+ A_{21}\boldsymbol{e}_2\otimes\boldsymbol{e}_1 + A_{22}\boldsymbol{e}_2\otimes\boldsymbol{e}_2 + A_{23}\boldsymbol{e}_2\otimes\boldsymbol{e}_3\\
&+ A_{31}\boldsymbol{e}_3\otimes\boldsymbol{e}_1 + A_{32}\boldsymbol{e}_3\otimes\boldsymbol{e}_2 + A_{33}\boldsymbol{e}_3\otimes\boldsymbol{e}_3
\end{aligned}$$

Herausfiltern einer Koordinate A_{ij}

$$A_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{A}\cdot\boldsymbol{e}_j = \boldsymbol{e}_i\cdot(A_{kl}\,\boldsymbol{e}_k\otimes\boldsymbol{e}_l)\cdot\boldsymbol{e}_j = \delta_{ik}\,A_{kl}\,\delta_{lj} = A_{ij}$$

damit gilt

$$\frac{\partial\Phi(\boldsymbol{A})}{\partial\boldsymbol{A}} = \frac{\partial\Phi}{\partial A_{ij}}\,\frac{\partial A_{ij}}{\partial\boldsymbol{A}} = \frac{\partial\Phi}{\partial A_{ij}}\,\frac{\partial\boldsymbol{e}_i\cdot\boldsymbol{A}\cdot\boldsymbol{e}_j}{\partial\boldsymbol{A}} = \frac{\partial\Phi}{\partial A_{ij}}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

$$\begin{aligned}\frac{\partial\Phi(\boldsymbol{A})}{\partial\boldsymbol{A}} &= \frac{\partial\Phi}{\partial A_{11}}\boldsymbol{e}_1\otimes\boldsymbol{e}_1+\frac{\partial\Phi}{\partial A_{12}}\boldsymbol{e}_1\otimes\boldsymbol{e}_2+\frac{\partial\Phi}{\partial A_{13}}\boldsymbol{e}_1\otimes\boldsymbol{e}_3\\ &+ \frac{\partial\Phi}{\partial A_{21}}\boldsymbol{e}_2\otimes\boldsymbol{e}_1+\frac{\partial\Phi}{\partial A_{22}}\boldsymbol{e}_2\otimes\boldsymbol{e}_2+\frac{\partial\Phi}{\partial A_{23}}\boldsymbol{e}_2\otimes\boldsymbol{e}_3\\ &+ \frac{\partial\Phi}{\partial A_{31}}\boldsymbol{e}_3\otimes\boldsymbol{e}_1+\frac{\partial\Phi}{\partial A_{32}}\boldsymbol{e}_3\otimes\boldsymbol{e}_2+\frac{\partial\Phi}{\partial A_{33}}\boldsymbol{e}_3\otimes\boldsymbol{e}_3\end{aligned}$$

Beispiel: Ableitung der ersten Invarianten I_A nach dem Tensor A

$$I_A = \mathrm{tr}\boldsymbol{A} = A_{11}+A_{22}+A_{33}$$

$$\frac{\partial I_A(\boldsymbol{A})}{\partial\boldsymbol{A}} = \left[\frac{\partial I_A}{\partial A_{ij}}\right]\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

$$\left[\frac{\partial I_A}{\partial A_{ij}}\right] = \begin{bmatrix}\frac{\partial\Phi}{\partial A_{11}} & \frac{\partial\Phi}{\partial A_{12}} & \frac{\partial\Phi}{\partial A_{13}}\\ \frac{\partial\Phi}{\partial A_{21}} & \frac{\partial\Phi}{\partial A_{22}} & \frac{\partial\Phi}{\partial A_{23}}\\ \frac{\partial\Phi}{\partial A_{31}} & \frac{\partial\Phi}{\partial A_{32}} & \frac{\partial\Phi}{\partial A_{33}}\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

also

$$\underline{\underline{\frac{\partial I_A(\boldsymbol{A})}{\partial\boldsymbol{A}} = \boldsymbol{I}}}$$

Beispiel: Ableitung eines Tensors A 'nach sich selbst'

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$$

$$\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{A}} = \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right]\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l \qquad \left[\frac{\partial A_{ij}}{\partial A_{kl}}\right] = \delta_{ik}\delta_{jl}$$

also

$$\underline{\underline{\frac{\partial\boldsymbol{A}}{\partial\boldsymbol{A}} = \boldsymbol{I}\overline{\otimes}\boldsymbol{I} = \mathbb{I}}}$$

Beispiel: Ableitung der Inversen A^{-1} nach dem Tensor A

$$\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} = \left[\frac{\partial A^{-1}_{ij}}{\partial A_{kl}}\right] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l$$

Überlegung

$$\frac{\partial \delta_{im}}{\partial A_{kl}} = \frac{\partial A^{-1}_{ij} A_{jm}}{\partial A_{kl}} = \frac{\partial A^{-1}_{ij}}{\partial A_{kl}} A_{jm} + A^{-1}_{ij} \frac{\partial A_{jm}}{\partial A_{kl}}$$

und damit

$$\left[\frac{\partial A^{-1}_{ij}}{\partial A_{kl}}\right] = -A^{-1}_{ij} \frac{\partial A_{jm}}{\partial A_{kl}} A^{-1}_{mj} = -A^{-1}_{ij} \delta_{jk} \delta_{ml} A^{-t}_{jm} = -A^{-1}_{ik}\, A^{-t}_{jl}$$

also

$$\underline{\underline{\frac{\partial \boldsymbol{A}^{-1}}{\partial \boldsymbol{A}} = -\boldsymbol{A}^{-1} \overline{\otimes} \boldsymbol{A}^{-t}}}$$

A.4 Verzerrungs– und Spannungsvektor

Aufgabe: In einer beliebigen Schnittebene charakterisiert durch die Normale $\boldsymbol{n}$ seien folgende (Proportionalitäts-)Beziehungen gegeben

$$\sigma_n = E_n\,\epsilon_n \qquad \boldsymbol{\sigma}_t = E_t\,\boldsymbol{\epsilon}_t$$

Bestimmen Sie den Spannungstensor $\boldsymbol{\sigma}^t$ für einen gegebenen Verzerrungstensor $\boldsymbol{\epsilon} = \nabla^{\text{sym}}\boldsymbol{u}$

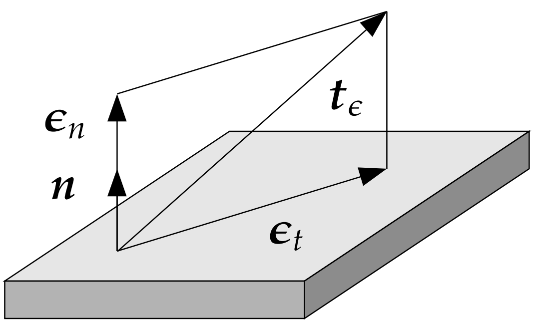
Benutzen Sie hierbei die folgenden Integralbeziehungen zur Integration des null-, zwei- und vierstufigen Fabric Tensors über den Vollwinkel Ω (Einheitskugel):

$$\begin{aligned}
\frac{3}{4\pi}\int_\Omega \quad & \mathrm{d}\Omega &= \quad & 3 \\
\frac{3}{4\pi}\int_\Omega \boldsymbol{n}\otimes\boldsymbol{n} \quad & \mathrm{d}\Omega &= \quad & \boldsymbol{I} \\
\frac{3}{4\pi}\int_\Omega \boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n} \quad & \mathrm{d}\Omega &= \quad & \frac{3}{5}\,\mathbb{I}^{\text{vol}} + \frac{2}{5}\,\mathbb{I}^{\text{sym}}
\end{aligned}$$

Bestimmung des Verzerrungsvektors t_ϵ der Ebene n – Projektion

Projektion des Verzerrungstensors $\boldsymbol{\epsilon}$ auf Verzerrungsvektor $\boldsymbol{t}_\epsilon$ der Ebene $\boldsymbol{n}$

$$\boldsymbol{t}_\epsilon(\boldsymbol{n}) = \boldsymbol{\epsilon}\cdot\boldsymbol{n} = \epsilon_n\,\boldsymbol{n} + \boldsymbol{\epsilon}_t \tag{A.4.1}$$



Normalverzerrungen (Skalar) -Dehnungen in Richtung von $\boldsymbol{n}$

$$\epsilon_n = \boldsymbol{t}_\epsilon \cdot \boldsymbol{n} = [\boldsymbol{\epsilon} \cdot \boldsymbol{n}] \cdot \boldsymbol{n} = \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\epsilon} : \boldsymbol{N} \tag{A.4.2}$$

Tangentialverzerrungen (Vektor) – Schubverzerrungen in der Ebene mit Normalenvektor $\boldsymbol{n}$

$$\begin{aligned} \boldsymbol{\epsilon}_t &= \boldsymbol{t}_\epsilon - \boldsymbol{\epsilon}_n = \boldsymbol{\epsilon} \cdot \boldsymbol{n} - \boldsymbol{\epsilon} : [\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] \\ &= \boldsymbol{\epsilon} : [\mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}] = \boldsymbol{\epsilon} : \boldsymbol{T} \end{aligned} \tag{A.4.3}$$

mit zweistufigem normalem Projektionstensor $\boldsymbol{N}$ und dreistufigem tangentialem Projektionstensor $\boldsymbol{T}$

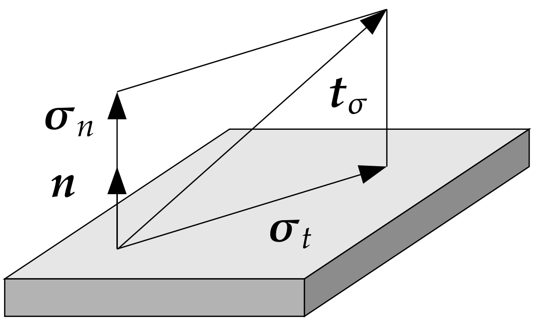
$$\begin{aligned} \boldsymbol{N}(\boldsymbol{n}) &= \boldsymbol{n} \otimes \boldsymbol{n} \\ \boldsymbol{T}(\boldsymbol{n}) &= \mathbb{I}^{\mathrm{sym}} \cdot \boldsymbol{n} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \end{aligned}$$

Indexdarstellung

$$\begin{aligned} \boldsymbol{N} &= N_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = n_i n_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ \boldsymbol{T} &= T_{ijm}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_m = [\tfrac{1}{2}\delta_{im} n_j + \tfrac{1}{2}\delta_{jm} n_i - n_i n_j n_m]\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_m \\ \boldsymbol{T}^{\mathrm{t}} &= T_{mij}\, \boldsymbol{e}_m \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j = [\tfrac{1}{2} n_i \delta_{jm} + \tfrac{1}{2} n_j \delta_{im} - n_i n_j n_m]\, \boldsymbol{e}_m \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{aligned}$$

Bestimmung des Spannungsvektors t_σ der Ebene n – Stoffgesetz

Betrachtung der Ebene mit Normale $\boldsymbol{n}$



linear elastisches Stoffgesetz für Normal- und Tangentialkomponente (Proportionalitätsbeziehung)

$$\sigma_n = E_n\, \epsilon_n \qquad \boldsymbol{\sigma}_t = E_t\, \boldsymbol{\epsilon}_t$$

mit Spannungsvektor $\boldsymbol{t}_\sigma$ der Ebene $\boldsymbol{n}$

$$\boldsymbol{t}_\sigma(\boldsymbol{n}) \;=\; \sigma_n\,\boldsymbol{n} + \boldsymbol{\sigma}_t \;=\; E_n\epsilon_n\,\boldsymbol{n} + E_t\boldsymbol{\epsilon}_t$$

Einsetzen der Kinematik liefert Beziehung zwischen Spannungsvektor $\boldsymbol{t}_\sigma$ und Verzerrungsvektor $\boldsymbol{t}_\epsilon$

$$\begin{aligned}\boldsymbol{t}_\sigma \;&=\; E_n[\boldsymbol{t}_\epsilon\cdot\boldsymbol{n}]\,\boldsymbol{n} \;+\; E_t[\boldsymbol{t}_\epsilon - [\boldsymbol{t}_\epsilon\cdot\boldsymbol{n}]\,\boldsymbol{n}\,] \\ &= [\,E_n[\boldsymbol{n}\otimes\boldsymbol{n}] \;+\; E_t[\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}]\;\;]\cdot\boldsymbol{t}_\epsilon\end{aligned}$$

Klammerausdruck definiert zweistufigen linear elastischen Materialoperator $\boldsymbol{E}(\boldsymbol{n})$ der Ebene $\boldsymbol{n}$

$$\boldsymbol{t}_\sigma = \boldsymbol{E}\cdot\boldsymbol{t}_\epsilon \qquad \text{mit} \qquad \boldsymbol{E} = E_n[\boldsymbol{n}\otimes\boldsymbol{n}] + E_t[\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n}]$$

Bestimmung des Spannungtensors σ^{t} – Integration

Definition des Cauchy Spannungstensors aus Spannungsvektoren $\boldsymbol{t}_{\sigma i}$ auf Ebenen parallel zu Koordinatenebenen

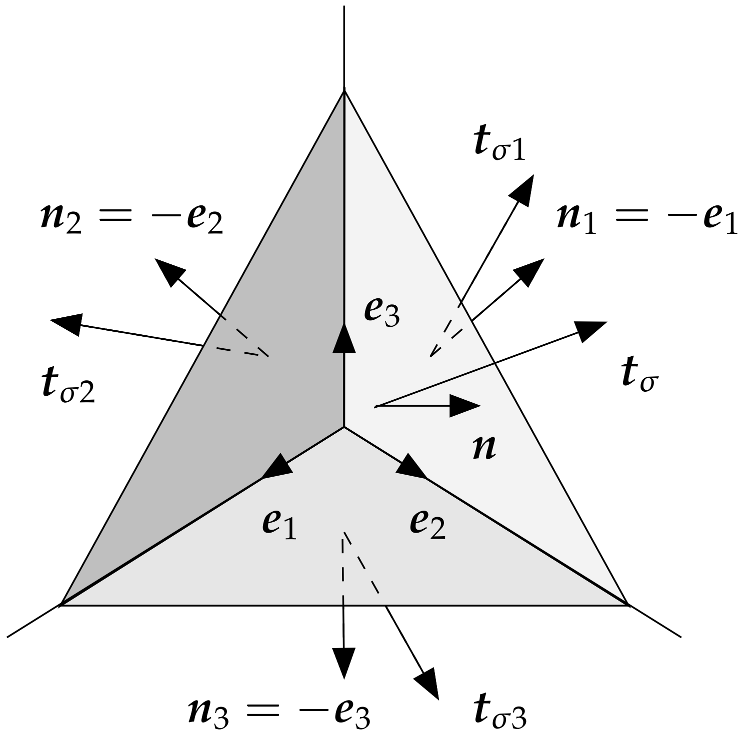
$$\boldsymbol{\sigma}^{\mathrm{t}} = \boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i = \sum_{i=1}^{n_{\mathrm{np}}}\boldsymbol{t}_{\sigma i}\otimes\boldsymbol{e}_i \qquad \text{mit} \qquad n_{\mathrm{np}} = 3$$

Interpretation mit Hilfe von Cauchy's Tetraeder
Verallgemeinerung mit Hilfe von Flächenanteilen

$$\frac{da_i}{da} = \cos\angle(\boldsymbol{e}_i,\boldsymbol{n}) = \boldsymbol{e}_i\cdot\boldsymbol{n} \qquad \boldsymbol{e}_i = \frac{da_i}{da}\,\boldsymbol{n}$$

Approximation des Spannungstensors $\boldsymbol{\sigma}$ aus Aufsummierung der Spannungsvektoren $\boldsymbol{t}_{\sigma i}$ von n_{np} belieibigen Flächen

$$\boldsymbol{\sigma}^{\mathrm{t}} = \sum_{i=1}^{n_{\mathrm{np}}}[\boldsymbol{t}_{\sigma i}\otimes\boldsymbol{n}_i]^{\mathrm{sym}}\frac{da_i}{da} \qquad \text{mit} \qquad n_{\mathrm{np}}\to\infty$$



für den Grenzübergang $n_{\mathrm{np}} \to \infty$ erhält man die Interpretation des Spannungstensors $\boldsymbol{\sigma}$ als Integration (aller) Spannungsvektoren $\boldsymbol{t}_\sigma(\boldsymbol{n})$ über den Vollwinkel Ω

$$\boldsymbol{\sigma}^{\mathrm{t}} = \tfrac{3}{4\pi} \int_\Omega [\boldsymbol{t}_\sigma \otimes \boldsymbol{n}]^{\mathrm{sym}} \mathrm{d}\Omega$$

mit Spannungsvektor $\boldsymbol{t}_\sigma(\boldsymbol{n}) = \sigma_n \, \boldsymbol{n} + \boldsymbol{\sigma}_t$

$$\boldsymbol{\sigma}^{\mathrm{t}} = \tfrac{3}{4\pi} \int_\Omega [\sigma_n \boldsymbol{n} \otimes \boldsymbol{n} + \boldsymbol{\sigma}_t \otimes \boldsymbol{n}]^{\mathrm{sym}} \mathrm{d}\Omega$$

weiterhin gilt (mit $\sigma_{tm} n_m = 0$)

$$\begin{aligned}
\boldsymbol{\sigma}_t \cdot \boldsymbol{T}^{\mathrm{t}} &= [\sigma_{tk} \boldsymbol{e}_k] \cdot [[\tfrac{1}{2} n_i \delta_{jm} + \tfrac{1}{2} n_j \delta_{im} - n_i n_j n_m] \boldsymbol{e}_m \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j] \\
&= [\tfrac{1}{2} \sigma_{tk} \delta_{km} n_i \delta_{jm} + \tfrac{1}{2} \sigma_{tk} \delta_{km} n_j \delta_{im} - \sigma_{tk} \delta_{km} n_i n_j n_m] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\
&= [\tfrac{1}{2} \sigma_{tj} n_i + \tfrac{1}{2} \sigma_{ti} n_j] \boldsymbol{e}_i \otimes \boldsymbol{e}_j = [\boldsymbol{\sigma}_t \otimes \boldsymbol{n}]^{\mathrm{sym}}
\end{aligned}$$

also

$$\boldsymbol{\sigma}^{\mathrm{t}} = \tfrac{3}{4\pi} \int_\Omega \sigma_n \boldsymbol{N} + \boldsymbol{\sigma}_t \cdot \boldsymbol{T}^{\mathrm{t}} \, \mathrm{d}\Omega$$

mit linear elastischem Stoffgesetz für Normal- und Tangentialkomponente

$$\begin{aligned} \sigma_n &= E_n \, \epsilon_n &= E_n \;\; \boldsymbol{\epsilon} : \boldsymbol{N} \\ \boldsymbol{\sigma}_t &= E_t \, \boldsymbol{\epsilon}_t &= E_t \;\; \boldsymbol{\epsilon} : \boldsymbol{T} \end{aligned}$$

gilt

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{t}} &= \tfrac{3}{4\pi} \int_\Omega E_n [\epsilon : \boldsymbol{N}] \; \boldsymbol{N} + E_t [\epsilon : \boldsymbol{T}] \cdot \boldsymbol{T}^{\mathrm{t}} \, \mathrm{d}\Omega \\ &= [E_n \tfrac{3}{4\pi} \int_\Omega \boldsymbol{N} \otimes \boldsymbol{N} \, \mathrm{d}\Omega + E_t \tfrac{3}{4\pi} \int_\Omega \boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{t}} \, \mathrm{d}\Omega] : \boldsymbol{\epsilon} \end{aligned}$$

analytische Integration des Normalanteils

$$\tfrac{3}{4\pi} \int_\Omega \boldsymbol{N} \otimes \boldsymbol{N} \mathrm{d}\Omega = \tfrac{3}{5} \mathbb{I}^{\mathrm{vol}} + \tfrac{2}{5} \, \mathbb{I}^{\mathrm{sym}} = \mathbb{I}^{\mathrm{vol}} + \tfrac{2}{5} \, \mathbb{I}^{\mathrm{dev}}$$

analytische Integration des Tangentialanteils
mit Indexdarstellung des Skalarproduktes $\boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{t}}$

$$\begin{aligned} \boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{t}} = & [\, \tfrac{1}{2} \delta_{im} n_j \quad + \tfrac{1}{2} \delta_{jm} n_i \quad - n_i n_j n_m \quad] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_m \\ & \cdot \; [\, \tfrac{1}{2} n_k \delta_{lm} \quad + \tfrac{1}{2} n_l \delta_{km} \quad - n_k n_l n_m \quad] \boldsymbol{e}_m \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \\ = & [\, \tfrac{1}{4} n_j n_k \delta_{il} \quad + \tfrac{1}{4} n_j n_l \delta_{ik} \quad - \tfrac{1}{2} n_i n_j n_k n_l \\ & + \; \tfrac{1}{4} n_i n_k \delta_{jl} \quad + \tfrac{1}{4} n_i n_l \delta_{jk} \quad - \tfrac{1}{2} n_i n_j n_k n_l \\ & - \; \tfrac{1}{2} n_i n_k n_k n_l - \tfrac{1}{2} n_i n_l n_k n_l + n_i n_j n_k n_l \quad] \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k \otimes \boldsymbol{e}_l \end{aligned}$$

$$\boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{t}} = \tfrac{1}{4} \boldsymbol{I} \underline{\otimes} \boldsymbol{N} + \tfrac{1}{4} \boldsymbol{I} \overline{\otimes} \boldsymbol{N} + \tfrac{1}{4} \boldsymbol{N} \overline{\otimes} \boldsymbol{I} + \tfrac{1}{4} \boldsymbol{N} \underline{\otimes} \boldsymbol{I} - \boldsymbol{N} \otimes \boldsymbol{N}$$

$$\begin{aligned} \tfrac{3}{4\pi} \int_\Omega \;\; \boldsymbol{T} \cdot \boldsymbol{T}^{\mathrm{t}} &= \tfrac{1}{4} \boldsymbol{I} \underline{\otimes} \boldsymbol{I} + \tfrac{1}{4} \boldsymbol{I} \overline{\otimes} \boldsymbol{I} + \tfrac{1}{4} \boldsymbol{I} \overline{\otimes} \boldsymbol{I} + \tfrac{1}{4} \boldsymbol{I} \underline{\otimes} \boldsymbol{I} - \tfrac{3}{5} \, \mathbb{I}^{\mathrm{vol}} - \tfrac{2}{5} \, \mathbb{I}^{\mathrm{sym}} \\ &= \mathbb{I}^{\mathrm{sym}} - \tfrac{3}{5} \, \mathbb{I}^{\mathrm{vol}} - \tfrac{2}{5} \, \mathbb{I}^{\mathrm{sym}} = \tfrac{3}{5} \, \mathbb{I}^{\mathrm{sym}} - \tfrac{3}{5} \, \mathbb{I}^{\mathrm{vol}} = \tfrac{3}{5} \, \mathbb{I}^{\mathrm{dev}} \end{aligned}$$

und damit

$$\boldsymbol{\sigma}^{\mathrm{t}} = [E_n \, [\mathbb{I}^{\mathrm{vol}} + \tfrac{2}{5} \, \mathbb{I}^{\mathrm{dev}}] + E_t \, [\tfrac{3}{5} \, \mathbb{I}^{\mathrm{dev}}]] : \boldsymbol{\epsilon}$$

oder alternativ

$$\boldsymbol{\sigma}^{\mathrm{t}} \;=\; [[E_n]\, \mathbb{I}^{\mathrm{vol}} + [\tfrac{2}{5}E_n + \tfrac{3}{5}E_t]\, \mathbb{I}^{\mathrm{dev}}] : \boldsymbol{\epsilon}$$

Klammerausdruck definiert vierstufigen linear elastischen Materialtensor $\mathbb{E}$

$$\boldsymbol{\sigma}^{\mathrm{t}} = \mathbb{E} : \boldsymbol{\epsilon} \qquad \text{mit} \qquad \mathbb{E} = [E_n]\, \mathbb{I}^{\mathrm{vol}} + [\tfrac{2}{5}E_n + \tfrac{3}{5}E_t]\, \mathbb{I}^{\mathrm{dev}}$$

später wird sich zeigen, dass für linear elastisches Hooke'sches Stoffgesetz gilt

$$\boldsymbol{\sigma}^{\mathrm{t}} = \mathbb{E} : \boldsymbol{\epsilon} \qquad \text{mit} \qquad \mathbb{E} = 3\,\kappa\, \mathbb{I}^{\mathrm{vol}} + 2\,\mu\, \mathbb{I}^{\mathrm{dev}}$$

mit Kompressionsmodul κ und Schubmodul μ, so dass

$$E_n = 3\,\kappa \qquad E_t = \tfrac{10}{3}\,\mu - 2\,\kappa$$