me 309
finite elements
in mechanical design

lecture notes, class 05
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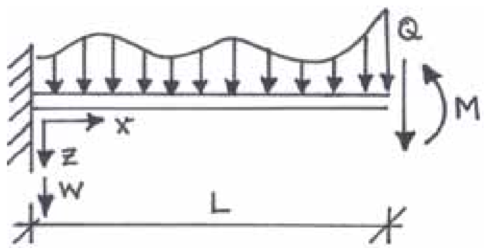
2 1d beam elements

2.1 Motivation

We have seen that finite element methods are ideally suited for second order partial differential equations! The beam equations we are trying to solve right now are differential equations of fourth order and thus a bit more complicated. In one dimension, however, there is an elegant way of solving fourth order differential equations by using a different set of shape functions for the interpolation of the unknown: Hermitian polynomials! We'll illustrate how they work in this section. Maybe you remember that there are different kinematic assumptions for beams, we will use the **Bernoulli beam** assumptions in the sequel.

2.2 Strong and weak form

Let's first look at the strong and weak form of beam equation. Similar to the 1d bar chapter, we start by summarizing the set of governing equations which consists of the followi-



ng three equations, the equilibrium, kinematics and constitutive equation.

equilibrium	$M''(x) - q(x) = 0$
kinematics	$\kappa(x) = -w(x)''$
const. eqn	$M(x) = EI\,\kappa(x)$

with

$w\ (x)$	...	transverse displacement (unknown)
$q\ (x)$	...	transverse line load
$M\ (x)$	...	bending moment
$Q\ (x)$	...	transverse force
$\kappa\ (x)$	...	curvature
EI	...	bending stiffness
E	...	young's modulus
I	...	moment of inertia
L	...	beam length

If you compare these equations to the 1d bar equations, you'll realize that they have the same format except for that fact that first order derivatives have been replaced by **second order** ones. To derive the element stiffness matrix and element load vector we will first look at the differential form that follows from combining all three equations above.

differential (strong) form find $w = w(x)$, such that

$EI\,w''''(x) - q$	(x)	$=$	0	$0 \le x \le L$	
w	(0)	$=$	0	on $\partial\mathcal{B}_w$	Dirichlet RB
w'	(0)	$=$	0	on $\partial\mathcal{B}_{w'}$	Dirichlet RB
$-EI\,w''$	(L)	$=$	$\bar{M}$	on $\partial\mathcal{B}_q$	Neumann RB
$-EI\,w'''$	(L)	$=$	$\bar{Q}$	on $\partial\mathcal{B}_m$	Neumann RB

In words, the differential or strong form states that we want to find a deflection field $w = w(x)$ that satisfies the partial differential equation $EI\,w''''(x) + q(x) = 0$ in the entire domain $0 < x < L$ and also satisfies the boundary conditions $w(0) = 0$ and $w'(0) = 0$ on the left and $-EI\,w''(L) = \bar{M}$ and $-EI\,w'''(L) = \bar{Q}$ on the right boundary.

The equation to be solved is a linear **fourth order partial differential equation**. Finite element methods like fourth order equations better than first or third order ones, but not as much as the second order equations! :-(

For beams, we want to be able to prescribe both, the deflection w and the angle w' as **Dirichlet boundary conditions**, and the transverse force $\bar{Q}$ and bending moment $\bar{M}$ as **Neumann boundary conditions**.

Similar to the 1d bar equation, we need to modify the 1d beam equation to make it solvable with finite element methods. We multiply it with a test function $v(x)$ and integrate it over the entire domain to obtain the

variational form find $w = w(x) \in H^{\text{trial}}$, such that

$$\boxed{\begin{array}{c} \displaystyle\int_0^L v(x)\,[\,EI\,w''''(x) - q(x)\,]\,\mathrm{d}x = 0 \qquad \forall\, v \in H_0^{\text{test}} \\ \ldots + \text{RB} \ldots \end{array}}$$

The variational form states that we want to find a deflection $w(x) \in H^{\text{trial}}$ in the set of **trial functions** for which the integral equation $\int_0^L v(x)\,[\,EI\,w''''(x) - q(x)\,]\,\mathrm{d}x = 0$ is satisfied for all possible **test functions** $v \in H_0^{\text{test}}$ in the set of test functions.

The variational form is obviously non-symmetric in v und w''''! So, similar to the 1d bar, we can modify it through an **integration by parts**, the Green Gauss theorem and the inclusion of the Neumann boundary conditions! And the really cool thing is that now that you know how to do it,

you can even do it **twice!** ;-)

$$\begin{aligned} \int_0^L v\,EI\,w''''\,\mathrm{d}x &= [v\,EI\,w''']_0^L - \int_0^L v^I EIw'''\,\mathrm{d}x \\ \int_0^L v^I\,EI\,w'''\,\mathrm{d}x &= \left[v^I\,EI\,w''\right]_0^L - \int_0^L v^{II} EIw''\,\mathrm{d}x \end{aligned} \tag{2.1}$$

Again, just remember the chain rule $[\,v\,w'''\,]' = v'\,u''' + v\,u''''$ and $[\,v'\,w''\,]' = v''\,u'' + v'\,u'''$! Altogether we can rewrite our variational equation

$$\begin{aligned} &\int_0^L v'' EIw''\mathrm{d}x - \int_0^L v\,q\,\mathrm{d}x \\ &+v(L)\,EI\,w'''(L) - v(0)\,EI\,w'''(0) \\ &-v^I(L)\,EI\,w''(L) + v^I(0)\,EI\,w''(0) = 0 \end{aligned} \tag{2.2}$$

with homogeneous (zero!) Dirichlet boundary conditions for the test functions $v(0) = 0$ und $v'(0) = 0$ and Neumann boundary conditions for $-EI\,w''(L) = \bar{M}$ and $EI\,w'''(L) = -\bar{Q}$.

Weak form find $w = w(x) \in H^2$, such that

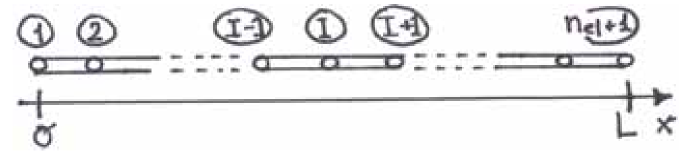
$$\boxed{\begin{array}{c} \displaystyle\int_0^L v'' EI\,w''\,\mathrm{d}x - \int_0^L v\,q\,\mathrm{d}x \\ -v(L)\,\bar{Q} + v^I(L)\,\bar{M} = 0 \quad \forall\, v \in H_0^2 \\ w(0) = 0 \quad \text{on } \partial\mathcal{B}_w \quad \text{Dirichlet RB} \\ w'(0) = 0 \quad \text{on } \partial\mathcal{B}_{w'} \quad \text{Dirichlet RB} \end{array}}$$

The weak form displays the typical format of a finite element equation: a symmetric term in terms of the two second order derivatives v'' and w'', another usually negative integral term that contains the **volume forces** such as gravity and the **boundary forces** that contain all external forces. The first one will become the stiffness matrix while the remaining terms will constitute the load vector.

Recall that from a mathematical point of view, we have relaxed the continuity requirements for the solution, that's why the symmetric variational form is also referred to as the weak form. But now, here is what is really more complicated than in bars: Mathematically speaking, our test and trial functions are now in H^2, i.e., for $w(x) \in H^2$ we require that $\int_0^L [w''(x)]^2 \, \mathrm{d}x < \infty$ exists! Basically, this states that unlike for bars, for beam elements the **second derivatives** have to be integrable! From a finite element point of view, this means that the solutions have to be somewhat smoother and not only the solution w itself but also its derivative w' has to be interpolated in a continuous way with no jumps across the inter-element boundaries!

2.3 Finite element discretization

discretization Let's discretize the domain $0 \leq x \leq L$ and divide it into a finite number of subdomains, the finite elements Ω_e with $e = 1, 2, ..., n_{\text{el}}$.



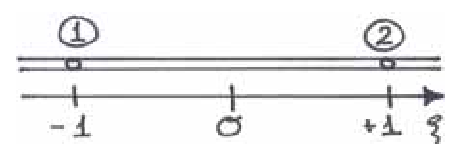
Then, the overall integral expressions (...) can be rewritten in terms of the individual element contibutions.

$$\int_0^L (...) \, \mathrm{d}x = \sum_{I}^{n_{\text{el}}} \int_{x_I}^{x_{I+1}} (...) \, \mathrm{d}x = \mathop{\mathsf{A}}_{e=1}^{n_{\text{el}}} (...) \tag{2.3}$$

To evaluate the individual element contributions to the global integrals, we will look at the element level now and elaborate a generic reference element in the isoparametric space.

2.4 Isoparametric concept

Again, we consider a generic reference element Referenzelement Ω_e with isoparametric coordinates ξ running from $\xi = -1$ to $\xi = +1$. The relations between the local and



global coordinates ξ and x are similar to the previous considerations for the 1d bar element.

$$\xi(x) = \frac{2x - [x_{I+1} + x_I]}{L_I} \quad \text{with} \quad L_I = x_{I+1} - x_I \tag{2.4}$$

with

$$\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{2}{L_I} \quad \text{or} \quad \mathrm{d}x = \frac{L_I}{2}\,\mathrm{d}\xi \tag{2.5}$$

for the transformation of the integrands. The fundamental difference to the 1d bar is that for the 1d beam, the discrete finite element solution has to be $w^h \in H^2$, i.e., its **second derivatives** have to be square integrable!

FE approximation on element level

Based on this requirement, we can introduce the finite element approximation for the trial functions on the element level

$$\boxed{\begin{aligned} w^h(\xi) &= \sum_{I=1}^{2} w_I N_I^0(\xi) + \sum_{I=1}^{2} w'_I N_I^1(\xi) \\ &= w_1 N_1^0(\xi) + w_2 N_2^0(\xi) \\ &+ w_1 N_1^1(\xi)\frac{L_e}{2} + w_2 N_2^1(\xi)\frac{L_e}{2} \end{aligned}}$$

and the finite element approximation for the test functions on the element level.

$$\boxed{\begin{aligned} v^h(\xi) &= \sum_{I=1}^{2} v_I N_I^0(\xi) + \sum_{I=1}^{2} v'_I N_I^1(\xi) \\ &= v_1 N_1^0(\xi) + v_2 N_2^0(\xi) \\ &+ v_1 N_1^1(\xi)\frac{L_e}{2} + v_2 N_2^1(\xi)\frac{L_e}{2} \end{aligned}}$$

If you compare this approximation to the one made for the 1d beam, you'll realize that we now introduce **two degrees of freedom** for each node, the deflection w_I and the angle w'_I. The element shape functions $N_1^0(\xi), N_2^0(\xi), N_1^1(\xi), N_2^1(\xi)$ look a bit different than before and they have an extra index. They are called Hermitian polynomials and we will explain them in the sequel.

2.5 Hermitian polynomials

from wiki In mathematics, the **Hermite polynomials** are a classical orthogonal polynomial sequence that arise in probability; in combinatorics, as an example of an Appell sequence, obeying the umbral calculus; and in physics, as the eigenstates of the quantum harmonic oscillator. They are named in honor of Charles Hermite.

This explanation of Hermitian polynomials looks way more complicated and mathematical then the wiki page for Lagrangian polynomials that we had used before. But, don't freak out, they are not that complicated after all! ;-) In general, Herimitian polynomials display the following property.

$$\begin{aligned} N_I^0\ (\xi_J) &= \delta_{IJ} & \frac{\mathrm{d}N_I^0(\xi_J)}{\mathrm{d}\xi} &= 0 \\ N_I^1\ (\xi_J) &= 0 & \frac{\mathrm{d}N_I^1(\xi_J)}{\mathrm{d}\xi} &= \delta_{IJ} \end{aligned} \tag{2.6}$$

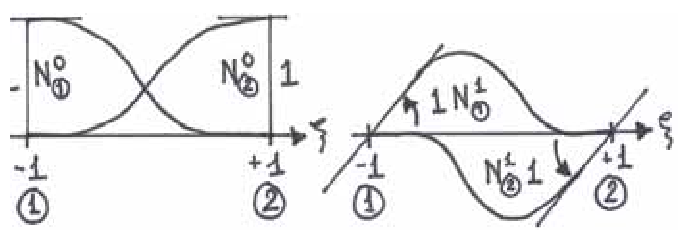
Similar to the Lagrangian polynomials, the Hermitian polynomials with an uppercase 0 are $N_I^0 = 1$ at node I and $N_I^0 = 0$ at the other node. Their **derivative** is zero $\mathrm{d}N_I^0(\xi_J) \,/\, \mathrm{d}\xi = 0$ at **both** nodes. In addition to the two N_I^0 shape functions, there are two shape functions N_I^1 indicated by an uppercase 1. Their **derivative** is $\mathrm{d}N_I^1(\xi_J) \,/\, \mathrm{d}\xi = 1$ at node I and $\mathrm{d}N_I^1(\xi_J) \,/\, \mathrm{d}\xi = 0$ at the other node. Their values are $N_I^1 = 0$ at **both** nodes.

To gain a better understanding of their shape, let's look at cubic Hermitian polynomials that are typically used in finite element analysis of fourth order problems.

$$
\begin{aligned}
N_1^0 \;(\xi) &= \tfrac{1}{4}\;[\,2 - 3\xi + \xi^3\,] \\
N_2^0 \;(\xi) &= \tfrac{1}{4}\;[\,2 + 3\xi - \xi^3\,] \\
N_1^1 \;(\xi) &= \tfrac{1}{4}\;[\,1 - \xi - \xi^2 + \xi^3\,] \\
N_2^1 \;(\xi) &= \tfrac{1}{4}\;[\,-1 - \xi + \xi^2 + \xi^3\,]
\end{aligned}
\qquad (2.7)
$$

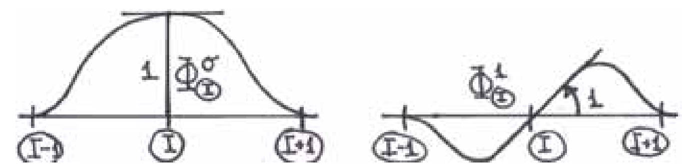
Hermitian polynomials on the element level (element e)

N_1^0 and N_2^0 $\qquad$ N_1^1 and N_2^1



Hermitian polynomials on the global level (node I)

continuity of function Φ_I $\qquad$ continuity of first derivative Φ_I'



You should remember that finite elements that use Hermitian polynomials **have no internal nodes**! The order of the interpolation can be increased by interpolating higher order derivatives, not by adding internal nodes as we have done for Lagrangian polynomials!

For the finite element implementation, it proves convenient to rearrange the approximation of the trial and test function and rewrite it in matrix notation since coding is easier and more computer oriented with matrices and vectors.

Approximation of solution on element level

$$w^h(\xi) = \mathbf{N}(\xi) \cdot \boldsymbol{w} \qquad (2.8)$$

Here, we have introduce the matrix $\mathbf{N}$ collecting all the shape functions

$$\mathbf{N}(\xi) = [\,N_1^0(\xi), N_1^1(\xi)\frac{L_e}{2}, N_2^0(\xi), N_2^1(\xi)\frac{L_e}{2}\,] \qquad (2.9)$$

and element vector of the nodal degrees of freedom $\boldsymbol{w}$.

$$\boldsymbol{w} = [\,w_1, w_1', w_2, w_2'\,] \qquad (2.10)$$

Approximation of test functions on element level

$$v^h(\xi) = \mathbf{N}(\xi) \cdot \boldsymbol{v} = \boldsymbol{v}^{\mathrm{t}} \cdot \mathbf{N}^{\mathrm{t}}(\xi) \tag{2.11}$$

With that, we'd now have to interpolate the second derivative of the deflection $w''(\xi)$ or rather the curvature $\kappa^h(\xi) = w''(\xi)$.

$$w''(\xi) = \frac{\mathrm{d}w^h(\xi)}{\mathrm{d}x^2} \tag{2.12}$$

Similar to the bar elements, we have to apply the chain rule to evaluate the derivative with respect to the physical coordinate x!

$$w''(\xi) = \frac{\mathrm{d}w^h(\xi)}{\mathrm{d}\xi^2}\frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} \tag{2.13}$$

By making use of

$$w^h(\xi) = \mathbf{N}(\xi) \cdot \boldsymbol{w} \qquad \text{and} \qquad \frac{\mathrm{d}^2\xi}{\mathrm{d}x^2} = \frac{4}{L_e^2} \tag{2.14}$$

we can rewrite the approximation of the curvature as follows,

$$w''(x) = \frac{4}{L_e}\frac{\mathrm{d}^2\mathbf{N}(\xi)}{\mathrm{d}\xi^2} \cdot \boldsymbol{w} = \mathbf{B}(\xi) \cdot \boldsymbol{w} \tag{2.15}$$

where we have introduced the discrete nodal operator $\mathbf{B}(\xi)$ which is referred to as **B-matrix** or **B-operator** in the finite element literature. In general, the B-matrix is a collection of

the derivatives of all shape functions $N(\xi)$ with respect to the physical coordinates x. For the 1d beam element, the B-matrix is actually a vector with the following entries.

$$\mathbf{B}^{\mathrm{t}} = \frac{4}{h_e^2}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\begin{bmatrix} N_1^0(\xi) \\ N_1^1(\xi)\,\frac{L_e}{2} \\ N_2^0(\xi) \\ N_2^1(\xi)\,\frac{L_e}{2} \end{bmatrix} = \frac{4}{L_e}\begin{bmatrix} +\frac{3}{2}\xi \\ \frac{1}{2}[-1+3\xi]\frac{L_e}{2} \\ -\frac{3}{2}\xi \\ \frac{1}{2}[+1+3\xi]\frac{L_e}{2} \end{bmatrix} \tag{2.16}$$

Note that the scaling factor $L_e\,/\,2$ has been introduced to keep units consistent since we deal with deflections w and their derivatives w' which are in fact angles.

The second derivative of the deflection, or rather the curvature, can thus be written in the following compact format.

Approximation of curvature on element level

$$\boxed{w''(x) = \mathbf{B}(\xi) \cdot \boldsymbol{w}}$$

$$\boxed{v''(x) = \mathbf{B}(\xi) \cdot \boldsymbol{v} = \boldsymbol{v}^{\mathrm{t}} \cdot \mathbf{B}^{\mathrm{t}}(\xi)}$$

That said, we can now use Hermitian polynomials to derive the stiffness matrix and the load vector for 1d beam elements. For this simple case, we can perform an analytical integration. Keep in mind, that the equations are generally not always that simple!

2.6 Analytical integration

analytical integration of element stiffness matrix

$$\int_{x_I}^{x_{I+1}} v_h''(x)\, EI\, w_h''(x)\,\mathrm{d}x = \int_{x_I}^{x_{I+1}} (\mathbf{B}\cdot\boldsymbol{v}) EI (\mathbf{B}\cdot\boldsymbol{w})\mathrm{d}x \tag{2.17}$$

By rearranging the terms, we can isolate $\boldsymbol{v}^{\mathrm{t}}$ and $\boldsymbol{w}$ from the integral expression

$$= \boldsymbol{v}^{\mathrm{t}} \cdot \underbrace{EI \frac{L_e}{2} \int_{\xi+-1}^{+1} \mathbf{B}^{\mathrm{t}}\,\mathbf{B}\mathrm{d}\xi}_{:=\mathbf{K}^e} \cdot \boldsymbol{w} \tag{2.18}$$

which then defines the element stiffness matrix.

stiffness matrix

$\mathbf{K} = \mathop{\mathsf{A}}\limits_{e=1}^{n_{\mathrm{el}}} \mathbf{K}^{\mathrm{e}}$	$\mathbf{K}^{\mathrm{e}} = EI\,\frac{h_e}{2}\int_{\xi=-1}^{\xi=+1}\mathbf{B}^{\mathrm{t}}\,\mathbf{B}\,\mathrm{d}\xi$
globale stiffness matrix	element stiffness matrix

Here, for our simple case, the element stiffness matrix can be integrated analytically.

$$\mathbf{K}^e = \frac{2\,EI}{L_e^3}\begin{bmatrix} +6 & +3L_e & -6 & +3L_e \\ +3L_e & +2L_e^2 & -3L_e & +L_e^2 \\ -6 & -3L_e & +6 & -3L_e \\ +3L_e & +L_e^2 & -3L_e & +2L_e^2 \end{bmatrix} \tag{2.19}$$

analytical integration of element load vector

$$\int_{x_I}^{x_{I+1}} v_h\, q(x)\,\mathrm{d}x = \int_{\xi=-1}^{+1} (\mathbf{N}^{\mathrm{t}}\cdot\boldsymbol{v})\, q(x(\xi))\frac{L_e}{2}\mathrm{d}\xi \tag{2.20}$$

Again, we rearrange the terms to isolate $\boldsymbol{v}^{\mathrm{t}}$ from the integral expression

$$\int_{x_I}^{x_{I+1}} v_h\, q(x)\,\mathrm{d}x = \boldsymbol{v}^{\mathrm{t}} \cdot \underbrace{\frac{L_e}{2}\int_{\xi+-1}^{+1} \mathbf{N}^{\mathrm{t}}\, q(x(\xi))\mathrm{d}\xi}_{:=\boldsymbol{F}^{\mathrm{e}}} \tag{2.21}$$

which defines the element load vector.

load vector

$\boldsymbol{F} = \mathop{\mathsf{A}}\limits_{e=1}^{n_{\mathrm{el}}} \mathbf{F}^{\mathrm{e}}$	$\boldsymbol{F}^{\mathrm{e}} = \frac{L_e}{2}\int_{\xi=-1}^{\xi=+1}\mathbf{N}^{\mathrm{t}}\,q\,\mathrm{d}\xi$
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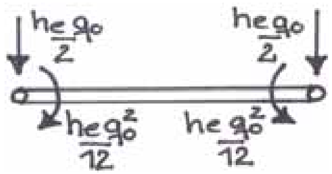
For simple cases, e.g. a constant transverse load $q = q_0$ =const, the element load vector can be integrated analytically.

$$\boldsymbol{F}^{\mathrm{e}} = \frac{L_e\,q_0}{2}\int_{\xi=-1}^{+1}\mathbf{N}^{\mathrm{t}}\,\mathrm{d}\xi \tag{2.22}$$

The nodal forces that are energetically equivalent to a constant transverse load take the following expression.

$$\boldsymbol{F}^{\mathrm{e}} = \frac{L_e\,q_0}{2}\left[1, +\frac{L_e}{6}, 1, -\frac{L_e}{6}\right]^{\mathrm{t}} \tag{2.23}$$



remark Remember that finite elements can only handle loads on their nodes! Line, are and volume loads have to be transformed into **energetically equivalent nodal forces** with the help of the element shape functions N_I^0 and N_I^1. For the Bernoulli beam element, this results in both transverse force and bending moment contributions!

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