1.5 Analytic integration

**Analytical integration of element stiffness matrix**

The figure shows the local element shape functions $N_1$ and $N_2$ on the left and the global functions $\Phi_I$ and $\Phi_{I+1}$ on the right.

$$\Phi_I(x) \quad \Phi_{I+1}(x)$$

Since our shape functions are linear, we are lucky and we can integrate the element stiffness matrix analytically.

$$K_{IJ} = \int_{\bar{\Omega}} \Phi'_I(x) EA \Phi'_J \, dx \quad (1.19)$$

of our bar element analytically. However, that is not always the case! Then, a numerical integration has to be performed, we will explain that later. Let’s write out the stiffness matrix in detail.

$$K_{IJ} = \int_{\bar{\Omega}} \frac{dN_I(\xi)}{dx} EA \frac{dN_J(\xi)}{dx} \, dx \quad I, J = 1, 2 \quad (1.20)$$

It’s important not to mix up the derivatives. Here, the derivatives are taken with respect to the global coordinates $x$. 
They have to be transformed to the local coordinates $\xi$, with the help of the chain rule.

$$\frac{dN_I(\xi)}{dx} = \frac{dN_I(\xi)}{d\xi} \frac{d\xi}{dx} = N'_I(\xi) \frac{d\xi}{dx} = N'_I(\xi) \frac{2}{L_e} \quad (1.21)$$

In some older finite element textbooks, you might come across the term **B-matrix**. That’s exactly what the above equation is about. The B-matrix contains the derivatives of the local element shape functions $N_I(\xi)$ with respect to the global coordinates $x$. Since $d\xi/dx = 2/L_e$ and thus $dx = L_e/2d\xi$, we end up with the following expression for the element stiffness matrix

$$K_{IJ} = \int_{\xi=-1}^{+1} N'_I(\xi) EA N'_J(\xi) \frac{2}{L_e} d\xi \quad I, J = 1, 2 \quad (1.22)$$

in terms of the linear element shape functions $N_I$ and their derivatives now with respect to the local isoparametric coordinates $N'_I$.

$$N_I(\xi) = \begin{cases} \frac{1}{2} [1 - \xi] & I = 1 \\ \frac{1}{2} [1 + \xi] & I = 2 \end{cases} \quad N'_I(\xi) = \begin{cases} -\frac{1}{2} & I = 1 \\ +\frac{1}{2} & I = 2 \end{cases} \quad (1.23)$$

All four entries of element stiffness matrix have to be integrated individually

$$K_{11} = \int_{\xi=-1}^{+1} \left[ -\frac{1}{2} \right] \left[ -\frac{1}{2} \right] \frac{2EA}{L_e} d\xi = \left[ +\frac{EA}{2L_e} \xi \right]_{\xi=-1}^{+1} = +\frac{EA}{L_e}$$

$$K_{12} = \int_{\xi=-1}^{+1} \left[ \frac{1}{2} \right] \left[ +\frac{1}{2} \right] \frac{2EA}{L_e} d\xi = \left[ -\frac{EA}{2L_e} \xi \right]_{\xi=-1}^{+1} = -\frac{EA}{L_e}$$

$$K_{21} = \int_{\xi=-1}^{+1} \left[ +\frac{1}{2} \right] \left[ -\frac{1}{2} \right] \frac{2EA}{L_e} d\xi = \left[ -\frac{EA}{2L_e} \xi \right]_{\xi=-1}^{+1} = -\frac{EA}{L_e}$$

$$K_{22} = \int_{\xi=-1}^{+1} \left[ +\frac{1}{2} \right] \left[ +\frac{1}{2} \right] \frac{2EA}{L_e} d\xi = \left[ +\frac{EA}{2L_e} \xi \right]_{\xi=-1}^{+1} = +\frac{EA}{L_e}$$

to yield the analytic expression for the element stiffness matrix for linear 1d bar elements.

$$K_{IJ} = \frac{EA}{L_e} \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix}$$

**analytical integration of element load vector**

Again, we can evaluate the integral expression analytically.

$$F_I = \int_{\xi=-1}^{+1} \Phi_I(x) f(x) d\xi$$

For now, let’s consider the example of a constant axial load $f(x) = f_0 = \text{const}$

$$F_I = \int_{\xi=-1}^{+1} N_I(\xi) f_0 \frac{L_e}{2} d\xi \quad (1.25)$$
with the linear shape functions \( N_I \).

\[
N_I(\xi) = \begin{cases} 
\frac{1}{2} [1 - \xi] & I = 1 \\
\frac{1}{2} [1 + \xi] & I = 2 
\end{cases}
\]  

(1.26)

The individual integration of the contributions to the element load vector renders the following expressions

\[
F_1^I = \int_{\xi=-1}^{+1} [1 - \xi] \frac{L_e}{4} f_0 d\xi = \frac{L_e}{4} f_0 \left[ \xi - \frac{1}{2} \xi^2 \right]_{\xi=-1}^{+1} = +\frac{1}{2} L_e f_0
\]

\[
F_2^I = \int_{\xi=-1}^{+1} [1 + \xi] \frac{L_e}{4} f_0 d\xi = \frac{L_e}{4} f_0 \left[ \xi + \frac{1}{2} \xi^2 \right]_{\xi=-1}^{+1} = +\frac{1}{2} L_e f_0
\]

which define the element load vector for linear bar elements under a constant line load.

\[
F_I^I = \frac{1}{2} L_e f_0 \begin{bmatrix} +1 \\ +1 \end{bmatrix}
\]

It’s important to keep in mind that loads can only be applied at the nodes and line, surface or volume loads have to be transformed into nodal loads with the above procedure.

**Assembly of element contributions**

Until now we have focused on the element level only. Now, let’s try to understand how all element contributions collectively build up the overall stiffness matrix. In the equations, we have denoted this process by the operator \( A_{e=1} \) which is referred to as **assembly** operator in the finite element literature. It symbolizes that the element contributions \( K_{ij}^I \) and \( F_i^I \) have to be added to the global stiffness matrix \( K \) and load vector \( F \). For this, we will need the connectivity table we have introduced earlier. It defines relations between the global and local coordinates. For our particular case, this is how the global system would look like.

\[
\begin{bmatrix}
K_{11} & K_{12} & 0 & 0 & 0 \\
K_{21} & K_{22} + K_{11} & K_{12} & 0 & 0 \\
0 & K_{21} & K_{22} + K_{11} & K_{12} & 0 \\
0 & 0 & K_{21} & K_{22} + K_{11} & K_{12} \\
0 & 0 & 0 & K_{21} & K_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5
\end{bmatrix}
= \begin{bmatrix}
F_1 \\
F_2 + F_1 \\
F_2 + F_1 \\
F_2 + F_1 \\
F_2
\end{bmatrix}
\]

Try to invert the matrix \( K \)! It’s not possible! We first have to build in the displacement boundary conditions \( u_1 = 0 \) and \( u_5 = 0 \) by basically crossing out their lines and columns. We are then left with a \([3 \times 3]\) system.

\[
U = K^{-1} \cdot F
\]  

(1.27)

Its solution defines the unknown nodal displacements \( u_2 \), \( u_3 \) and \( u_4 \). Note the band structure of the global stiffness matrix! That’s really handy when we’re solving large systems of equations and only have to store its non-zero entries! Again, \( K \) is symmetric and positive definite!
1.6 Boundary conditions

**Dirichlet boundary conditions**

The Dirichlet boundary conditions are sometimes also referred to as essential boundary conditions because they have to be prescribed at least at some points in the domain. They are the boundary conditions for the primary unknown, which, in our case, is the displacement $u$.

\[
\begin{align*}
  u(x = 0) &= \bar{u}_1 \quad \text{so far} \quad u(x = 0) = 0 \\
  u(x = L) &= \bar{u}_n \quad \text{so far} \quad u(x = L) = 0
\end{align*}
\] (1.28)

We have previously assumed the left and right boundary were fixed such that $u(x = 0) = 0$ and $u(x = L) = 0$, i.e., we have applied **homogeneous Dirichlet boundary conditions**. Accounting for non-zero displacements $u(0) \neq 0$ or $u(L) \neq 0$, i.e., prescribing **inhomogeneous Dirichlet boundary conditions** is somewhat complicated from a mathematical point of view since these then appear as entries in the vector of unknowns $U$! Check out the system of equations!

\[
\begin{bmatrix}
  K_{11} & K_{12} & 0 & 0 \\
  K_{21} & K_{22} & \ldots & 0 \\
  0 & \ldots & \ldots & 0 \\
  0 & 0 & \ldots & K_{n-1,n-1} K_{n-1,n}
\end{bmatrix}
\begin{bmatrix}
  \bar{u}_1 \\
  u_2 \\
  \ldots \\
  u_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
  F_1 \\
  F_2 \\
  \ldots \\
  F_{n-1}
\end{bmatrix}
\] (1.29)

Elimination of the first and last line yields a reduced system of equations with $[n-2]$ equations for the $[n-2]$ unknown nodal displacements $u_2, u_3, \ldots, u_{n-1}$. For homogeneous boundary conditions, this elimination is trivial. For inhomogeneous boundary conditions, however, the righthand side has to be modified accordingly. Try to understand where the $-K_{21} \bar{u}_1$ and $-K_{n-1,n} \bar{u}_n$ terms come from!

\[
\begin{bmatrix}
  K_{22} & K_{23} & 0 & 0 \\
  K_{32} & K_{33} & \ldots & 0 \\
  0 & \ldots & \ldots & 0 \\
  0 & 0 & \ldots & K_{n-2,n-2} K_{n-2,n-1}
\end{bmatrix}
\begin{bmatrix}
  u_2 \\
  u_3 \\
  \ldots \\
  u_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
  F_2 - K_{21} \bar{u}_1 \\
  F_3 \\
  \ldots \\
  F_{n-1} - K_{n-1,n} \bar{u}_n
\end{bmatrix}
\] (1.30)

The eliminated first and last line serve to determine the reaction forces $V_1$ and $V_n$ at the left and right boundary.

\[
\begin{align*}
  K_{11} & \quad u_1 \quad + \quad K_{12} \quad u_2 \quad = \quad F_1 \quad - \quad V_1 \\
  K_{n,n-1} u_{n-1} \quad + \quad K_{nn} \quad u_n \quad = \quad F_n \quad + \quad V_n
\end{align*}
\] (1.31)

**remark** The solution of the global system of equations requires the inversion of the global stiffness matrix. Without Dirichlet boundary conditions, i.e., without the elimination of lines and columns, the global stiffness matrix would be singular and thus non-invertible. That’s why Dirichlet boundary
conditions are called sometimes essential boundary conditions. The minimum number of Dirichlet boundary conditions corresponds to the number of rigid body modes, i.e., one in 1d, three in 2d and six in 3d.

**Neumann boundary conditions**

The Neumann boundary conditions are referred to as natural boundary conditions. They are boundary conditions for derivatives of the primary unknown $u'$ which, in our case, are related to the force $\bar{V} = EAu'$.

$$EAu' (x = 0) = \bar{V}_1 \quad \text{so far} \quad u (x = 0) = 0$$

$$u (x = L) = \bar{u}_n \quad \text{so far} \quad u (x = L) = 0 \quad (1.32)$$

The incorporation of homogeneous or inhomogeneous Neumann boundary conditions is relatively easy since they just appear on the right-hand side of the global system of equations. Since some Dirichlet boundary conditions have to be prescribed, we chose to prescribe the displacements on the right boundary $u (x = L) = \bar{u}_n$.

$$\begin{bmatrix}
K_{11} & K_{12} & 0 & 0 & 0 \\
K_{22} & K_{23} & 0 & 0 & 0 \\
0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & \ldots & K_{n-2,n-2} & K_{n-2,n-1} \\
0 & 0 & 0 & K_{n-1,n-2} & K_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
\vdots \\
u_{n-2} \\
u_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
F_1 - \bar{V}_1 \\
F_2 \\
\vdots \\
F_{n-2} \\
F_{n-1} - K_{n-1,n} \bar{u}_n
\end{bmatrix}$$

(1.33)

Again, we can use the eliminated equation of the right boundary node $n$ to determine the reaction force $V_n$ on the right boundary.

$$K_{n,n-1} u_{n-1} + K_{nn} u_n = F_n + V_n \quad (1.34)$$

**remark** Although we prescribe forces $\bar{V}_1$ on the left boundary, we still have to apply at least one Dirichlet boundary condition to fix the structure in space, avoid rigid body motion and ensure that the stiffness matrix is non-singular and thus invertible.

### 1.7 Stress calculation

When designing a structure, we are usually not interested in the displacements but rather in their derivatives, i.e., the strains or, more importantly, the stress.
The problem is that an elementwise interpolation with linear shape functions \( N_I \) yields a continuous (mathematically: \( C^0 \)-continuous) displacement field \( u_h \), see figure left. The derivatives of the shape functions \( N'_I \), however, are discontinuous resulting in jumps in the \( u''_h \) (mathematically: \( C^{-1} \)-continuous) and accordingly discontinuous strain \( \epsilon \) and stress fields \( \sigma(x) = EA u'(x) \), see figure right.

**Example** stress calculation in one dimensional bar

\[
\begin{align*}
\text{analytical solution} \\
u''(x) &= \frac{f}{EA} \\
\text{1st integration} \\
u'(x) &= \int_{w=0}^{w=x} \frac{f}{EA} \, dw + c_1 = \frac{f}{EA} \left. w \right|_{w=0}^{w=x} + c_1 \\
u'(x) &= \frac{f}{EA} x + c_1 \\
\text{2nd integration} \\
u(x) &= \int_{v=0}^{v=x} \frac{f}{EA} \, dv + c_1 x + c_2 = \frac{f}{2EA} \left. v^2 \right|_{v=0}^{v=x} + c_1 x + c_2 \\
\end{align*}
\] (1.35) (1.36) (1.37) (1.38)

boundary conditions

\[
\begin{align*}
u(0) &= 0 \quad c_2 = 0 \\
u(L) &= 0 \quad c_1 = -\frac{fL}{2EA} \\
\end{align*}
\] (1.40)

analytical solution for displacement field and derivative

\[
\begin{align*}
u(x) &= \frac{f}{EA} \left[ x^2 - xL \right] \\
u'(x) &= \frac{f}{EA} \left[ x - \frac{1}{2} L \right] \\
\end{align*}
\] (1.41) (1.42)

numerical solution

\[
\begin{align*}
u'(x) &= \begin{cases} 
-0.4 & e = 1 \\
-0.2 & e = 2 \\
0.0 & e = 3 \\
0.2 & e = 4 \\
0.4 & e = 5 
\end{cases} \\
\end{align*}
\] (1.43)

example \( u'_{e=2}(x) = \epsilon_3 - \epsilon_2 \)

\[
\begin{align*}
\frac{u_3 - u_2}{L_e} &= -0.04 \\
\frac{0.2}{0.2} &= -0.2 = \text{const} \\
\end{align*}
\]

The numerical solution is identical to the analytical solution in the element center points, these are called *super convergent* points.
accuracy of the solution / error analysis / adaptivity

The accuracy of the finite element approximation can be analyzed by means of an error function \( e(x) \), i.e., the difference of the exact solution \( u(x) \) and the numerical solution \( u^h(x) \).

\[
e(x) = u(x) - u^h(x) \tag{1.44}
\]

Since \( e(x) \) is a local measure, a reasonable measure to quantify the global overall error is the error norm \( ||e(x)|| \), a measure, e.g., in the \( L^2 \)-norm.

\[
||e(x)|| = \left[ \int_0^L e^2 \, dx \right]^{1/2} = \left[ \int_0^L [u(x) - u^h(x)]^2 \, dx \right]^{1/2} \tag{1.45}
\]

The convergence of the solution is the property of a decreasing error \( ||e(x)|| \to 0 \) as the number of elements increases \( n_d \to \infty \), i.e., the element size decreases \( L_e \to 0 \). The convergence upon mesh refinement is related to the notion of h-adaptivity. In the example of the one dimensional bar, the numerical solution \( u^h \) converges towards the exact (analytical) solution \( u \) for an increasing number of elements. An alternative strategy to improve the solution is to increase the polynomial order of the interpolation which is related to the notion p-adaptivity. We will discuss higher order polynomials to improve the solution next.

1.8 Lagrangian polynomials

from wiki In numerical analysis, a Lagrange polynomial, named after Joseph Louis Lagrange, is the interpolation polynomial for a given set of data points in the Lagrange form. It was first discovered by Edward Waring in 1779 and later rediscovered by Leonhard Euler in 1783.

Until now, we have used a linear interpolation for the unknown

\[
u^h(\xi) = \sum_{i=1}^{n_{nod}} u_i N_i(\xi) = u_1 N_1(\xi) + u_2 N_2(\xi) \tag{1.46}
\]

where the interpolation or shape functions \( N_1(\xi) \) and \( N_2(\xi) \) are Lagrangian polynomials. For two noded elements, they are piecewise linear.

\[
N_1(\xi) = \frac{1}{2} [1 - \xi] \quad N_2(\xi) = \frac{1}{2} [1 + \xi] \tag{1.47}
\]

Remember that \( \xi \) are the isoparametric coordinates which run from -1 to +1 on our isoparametric reference element. General Lagrangian polynomials of degree \( k \) can be constructed according to the following formula.

\[
N_i = \frac{[\xi - \xi_1][\xi - \xi_2][\xi - \xi_{i-1}][\xi - \xi_{i+1}]...[\xi - \xi_k+1]}{[\xi_i - \xi_1][\xi_i - \xi_2][\xi_i - \xi_{i-1}][\xi_i - \xi_{i+1}]...[\xi_i - \xi_k+1]} \]

example $k=1$, linear shape functions

Lagrangian polynomials of linear order are based on two nodes $\xi_1 = -1$ and $\xi_2 = +1$.

\[
N_1(\xi) = \frac{\xi - 1}{\xi_1 - \xi_2} = \frac{\xi - 1}{-1 - 1} = \frac{1}{2}(1 - \xi) \\
N_2(\xi) = \frac{\xi - \xi_2}{\xi_2 - \xi_1} = \frac{\xi + 1}{-1 + 1} = \frac{1}{2}(1 + \xi) \quad (1.48)
\]

According to the general construction formula for Lagrangian polynomials, $N_1$ and $N_2$ can then be derived as follows.

\[
N_1(\xi) = \begin{cases} 
1 & \xi = -1 \\
0 & \xi = +1
\end{cases}
\]

\[
N_2(\xi) = \begin{cases} 
1 & \xi = -1 \\
0 & \xi = +1
\end{cases}
\]

(1.49)

In agreement with the definition of Lagrangian polynomials, $N_2$ takes the value 1 at node $I = 2$ and 0 at all other nodes.

example $k=2$, quadratic shape functions

Lagrangian polynomials of quadratic order are based on three nodes $\xi_1 = -1$, $\xi_2 = 0$ and $\xi_3 = +1$.

\[
N_1(\xi) = \frac{[\xi - \xi_2][\xi - \xi_3]}{[\xi_1 - \xi_2][\xi_1 - \xi_3]} = \frac{\xi[\xi - 1]}{[-1 - 0][-1 + 1]} = \frac{1}{2}\xi[\xi - 1] \\
N_2(\xi) = \frac{[\xi_2 - \xi_1][\xi_2 - \xi_3]}{[\xi_2 - \xi_1][\xi_2 - \xi_3]} = \frac{\xi_2^2 - 1}{[0 + 1][0 - 1]} = 1 - \xi^2 \\
N_3(\xi) = \frac{[\xi_3 - \xi_1][\xi_3 - \xi_2]}{[\xi_3 - \xi_1][\xi_3 - \xi_2]} = \frac{\xi_3^2 + 1}{[1 - 0][-1]} = \frac{1}{2}\xi[\xi + 1] \quad (1.50)
\]

We can control their interpolation property, e.g., for $N_1(\xi)$.

\[
N_1(\xi = -1) = \frac{1}{2}[-1][-1 - 1] = 1 \quad \sqrt{ } \\
N_1(\xi = 0) = \frac{1}{2}[0][0 - 1] = 0 \quad \sqrt{ } \quad (1.51) \\
N_1(\xi = +1) = \frac{1}{2}[1][0 + 1] = 0 \quad \sqrt{ }
\]

According to the definition of Lagrangian polynomials, $N_1$ takes the value 1 at node $I = 1$ and 0 at all other nodes.
example quadratic bar elements

The general formula for the stiffness matrix of bar elements has been derived earlier. Remember, it looked somewhat like this!

\[
K_{IJ} = \int_{\Omega} \frac{d}{d\xi} N_I(\xi) \frac{d}{d\xi} N_J(\xi) \, d\xi
\]

\[
= \int_{\Omega} \frac{2}{L_e} \frac{d}{d\xi} N_I(\xi) \frac{2}{L_e} \frac{d}{d\xi} N_J(\xi) \frac{L_e}{2} \, d\xi
\]

\[
= \int_{\xi=-1}^{\xi=+1} \frac{2}{L_e} \frac{d}{d\xi} N_I(\xi) \frac{d}{d\xi} N_J(\xi) \, d\xi
\]

(1.52)

Rather than using the linear polynomials now, we evaluate the stiffness matrix in terms of the quadratic Lagrangian polynomials \(N_I\) and their derivatives \(dN_I / d\xi\).

\[
N_1 = \frac{1}{2} \xi [\xi - 1] \quad \frac{d}{d\xi} N_1 = \xi - \frac{1}{2}
\]

\[
N_2 = \xi^2 - 1 \quad \frac{d}{d\xi} N_2 = 2 \xi
\]

\[
N_3 = \frac{1}{2} \xi [\xi + 1] \quad \frac{d}{d\xi} N_3 = \xi + \frac{1}{2}
\]

Each component of the stiffness matrix can then be integrated analytically. For the entry \(K_{11}\), this evaluation reads as follows.

\[
K_{11} = \int_{\xi=-1}^{\xi=+1} \frac{2}{L_e} \frac{d}{d\xi} N_1(\xi) \frac{d}{d\xi} N_1(\xi) \, d\xi
\]

\[
= \int_{\xi=-1}^{\xi=+1} \frac{2}{L_e} \left[ \xi - \frac{1}{2} \right] \left[ \xi - \frac{1}{2} \right] \, d\xi
\]

(1.53)

All the other terms can be evaluated accordingly. Since the quadratic element has three nodes, there are nine entries in total and the element stiffness matrix for quadratic one-dimensional bar elements has the dimension \([3 \times 3]\).

\[
K_{IJ} = \frac{1}{3 L_e} \begin{bmatrix}
+7 & -8 & +1 \\
-8 & +16 & -8 \\
+1 & -8 & +7
\end{bmatrix}
\]

(1.55)

The element load vector can be evaluated in a similar way. Recall its general formula.

\[
F_I = \int_{\Omega} N_I(\xi) f(x) \, d\xi
\]

\[
= \int_{\xi=-1}^{\xi=+1} N_I(\xi) f(\xi) \frac{L_e}{2} \, d\xi
\]

(1.56)

For our example of a constant line load \(f(x) = f_0 = \text{const}\), it reduces to the following expression

\[
F_I = \frac{L_e f_0}{2} \int_{\xi=-1}^{\xi=+1} N_I(\xi) \, d\xi
\]

(1.57)

which can be evaluated in terms of the quadratic Lagrangian polynomials \(N_I\). The element load vector for quadratic
one dimensional bar elements under a constant line load has three entries, one for each node.

$$F_i^c = \frac{1}{6} L_c f_0 \begin{bmatrix} +1 \\ +4 \\ +1 \end{bmatrix}$$

![Load distribution diagram]

Surprisingly, the line load is not distributed equally on the three nodes. The boundary nodes receive 1/6 of the total load, the center node 4/6. Of course, the overall load adds up to one. When you run a commercial code like ANSYS, this redistribution of loads is done automatically by the program depending on the choice of elements and the order of interpolation. Keep in mind though, that the program internally converts line, area and volume loads to consistent nodal forces since finite elements can only handle loads on their nodes!

What we have done in this section is we tried to increase the order of accuracy of the order of our interpolation. The improvement of the solution by increasing the polynomial degree $p$ is referred to as p-adaptivity in the mathematical finite element literature.

**remark** Here’s the bad news though! No matter how hard you try to improve the solution by increasing the order of the interpolation, there will always be jumps in the derivative of the unknown $u^h$! The interpolation $u^h$ is only $C^0$-continuous within the element and the jumps at the interelement boundaries remain, no matter how much we improve the local interpolation on the element level! That’s the very nature of the finite element method and Lagrangian polynomials!

The figure displays an elementwise quadratic interpolation of with a continuous displacement field $u^h(x)$ on the left and a resulting elementwise linear derivative $u^{h'}(x)$ which results in finite element specific discontinuous strain and stress fields $\epsilon(x)$ and $\sigma(x)$. Because jumps are ugly, finite element programs usually smooth out discontinuous stress fields during the post-processing step and everybody’s happy! But... what you actually see is not the real solution, it’s the averaged nodal stresses!

In the next section, we will introduce beam elements with $C^1$-continuous Hermitian polynomials that provide a smoother interpolation of the solution.
1.9 Example

given \( F = 100 \text{kN}, f = 10 \text{ KN/cm}, E = 21000 \text{ kN/cm}^2, \ A_1 = 1 \times 1 \text{ cm}^2, A_2 = 0.6 \times 1 \text{ cm}^2 \)

\[ \begin{array}{c}
\text{EA}_1 \\
70\text{cm}
\end{array} \quad \begin{array}{c}
F \\
\end{array} \quad \begin{array}{c}
\text{EA}_2 \\
30\text{cm}
\end{array} \]

\[ f(x) \]

\text{problem I} discretize the above system with two quadratic bar elements!

\[ u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \]

(1) 2 3 4 5

\( n_{el} = 4 \) elements, \( n_{nod} = 5 \) nodes, number of degrees of freedom \( n_{dof} = 5 - 2 = 3 \)

\text{problem II} determine the element stiffness matrices of both elements!

general element stiffness matrix for quadratic one dimensional bar elements

\[ K_{ij} = \frac{EA}{3L_e} \begin{bmatrix} +7 & -8 & +1 \\ -8 & +16 & -8 \\ +1 & -8 & +7 \end{bmatrix} \]

for element 1: \( E A_1 = 21000 \) and \( L_1 = 70 \)

for element 2: \( E A_2 = 12600 \) and \( L_2 = 30 \)

\[ K_1^{ij} = \begin{bmatrix} +700 & -800 & +100 \\ -800 & +1600 & -800 \\ +100 & -800 & +700 \end{bmatrix} \]

\[ K_2^{ij} = \begin{bmatrix} +960 & -1120 & +140 \\ -1120 & +2240 & -1120 \\ +140 & -1120 & +980 \end{bmatrix} \]

\text{problem III} determine the element load vectors for both elements!

general element load vector for quadratic one dimensional bar elements under constant line load

\[ F_i = \frac{1}{6} L_e f_0 \begin{bmatrix} +1 & +4 & +1 \end{bmatrix} \]

for element 1: \( L_1 f_0 = 70 \)

for element 2: \( L_2 f_0 = 30 \)

\[ F_1^1 = \frac{700}{6} \begin{bmatrix} +1 \\ +4 \\ +1 \end{bmatrix} \]

\[ F_2^1 = \frac{300}{6} \begin{bmatrix} +1 \\ +4 \\ +1 \end{bmatrix} \]
**problem IV** determine the axial displacement at the interfa-
ced between the two bars!

assembly of element stiffness matrices to global stiffness ma-
trix

\[
K = \begin{bmatrix}
+700 & -800 & +100 & 0 & 0 \\
-800 & +1600 & -800 & 0 & 0 \\
+100 & -800 & +1680 & -1120 & +140 \\
0 & 0 & -1120 & +2240 & -1120 \\
0 & 0 & +140 & -1120 & +980 \\
\end{bmatrix}
\]  (1.61)

assembly of element load vectors to global load vector

\[
F = \frac{1}{3} \begin{bmatrix}
350 \\
1400 \\
500 \\
600 \\
150 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
100 \\
0 \\
0 \\
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
350 \\
1400 \\
800 \\
600 \\
150 \\
\end{bmatrix}
\]  (1.62)

global system of equations \( K \cdot U = F \) with Dirichlet bounda-
ry conditions

\[
\begin{bmatrix}
+1600 & -800 & 0 \\
-800 & +1680 & -1120 \\
0 & -1120 & +2240 \\
\end{bmatrix} \begin{bmatrix}
u_2 \\
u_3 \\
u_4 \\
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1400 \\
800 \\
600 \\
\end{bmatrix}
\]  (1.63)

solution

\[ u_2 = 0.7083cm \quad u_3 = 0.8333cm \quad u_4 = 0.5060cm \]

The axial displacement at the interface between the two bars is \( u_3 = 0.8333 \) cm.

**problem V** determine the normal forces in both elements
displacement field

\[ u(\xi) = \sum_{i=1}^{3} N_i(\xi) \, u_i \]  (1.64)

for element 1

\[ u^1(\xi) = \left[ \frac{1}{2} \xi^2 - \frac{1}{2} \xi \right] 0.0000cm + \left[ 1 - \xi^2 \right] 0.7083cm + \left[ \frac{1}{2} \xi^2 + \frac{1}{2} \xi \right] 0.8333cm \]

\[ u^1(\xi) = \left[ -0.2917\xi^2 + 0.4166\xi + 0.7083 \right] \] cm

for element 2

\[ u^2(\xi) = \left[ \frac{1}{2} \xi^2 - \frac{1}{2} \xi \right] 0.8333cm + \left[ 1 - \xi^2 \right] 0.5060cm + \left[ \frac{1}{2} \xi^2 + \frac{1}{2} \xi \right] 0.0000cm \]

\[ u^2(\xi) = \left[ -0.0894\xi^2 - 0.4166\xi + 0.5060 \right] \] cm
control of equations at nodes

\[ u^1(\xi = -1) = 0.0000 \text{ cm} \quad u^1(\xi = +1) = 0.8332 \text{ cm} \]
\[ u^2(\xi = -1) = 0.8332 \text{ cm} \quad u^2(\xi = +1) = 0.0000 \text{ cm} \]

normal forces

\[ N(\xi) = \sum_{I=1}^{3} EA \frac{dN_I(\xi)}{dx} u_I \]

for element 1

\[ N^1(\xi) = [250 - 350\xi] \text{ kN} \]

for element 2

\[ N^2(\xi) = [-350 - 150\xi] \text{ kN} \]

control of equations at nodes

\[ N^1(\xi = -1) = +600 \text{ kN} \quad N^1(\xi = +1) = -100 \text{ kN} \]
\[ N^2(\xi = -1) = -200 \text{ kN} \quad N^2(\xi = +1) = -500 \text{ kN} \]