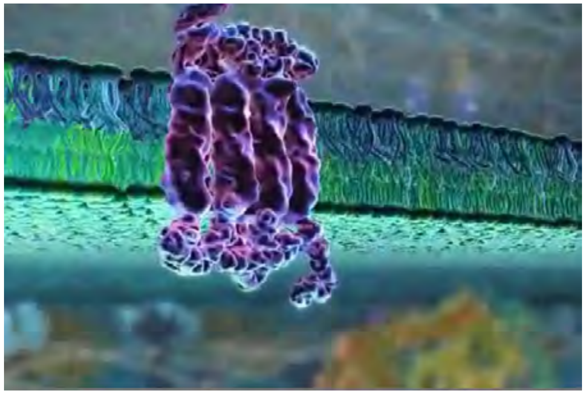
5.2 biomembranes - energy



the inner life of a cell, viel & lue, harvard [2006]

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the cell membrane

all cellular components are contained within a cell membrane which is **extremely thin**, approximately 4-5nm, and **very flexible**. inside the cell membrane, most cells behave like a liquid as they consist of more than 50% of water. the cell membrane is **semi-permeable** allowing for a controlled exchange between intracellular and extracellular components and information.

mechanisms of transport through the membrane

- passive transport driven by gradients in concentration
- active transport that does require extra energy; it is regulated by ion channels, pumps, transporters, exchangers and receptors

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the cell membrane

the barrier between the inner and outer cell is the cell membrane, a **bilayer** consisting of **phospholipids** of a characteristic structural arrangement. in aqueous solutions, these phospholipids essentially display two kinds of non-covalent interactions.

non-covalent interactions of phospholipids

- hydrophobic, water avoiding non-polar residues
- hydrophilic, water loving polar head groups

this behavior is similar to fatty acids or **oil in water**, where the hydrophilic polar heads tend to be oriented towards the water phase while the hydrophobic tails are oriented towards the oil phase.

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the lipid bilayer

Figure 5.1: Electron microscopy of the cell membrane stained with osmium tetroxide illustrating the polar head groups with a light 2nm space of hydrophobic tails sandwiched between them, adopted from [4]

the cell membrane

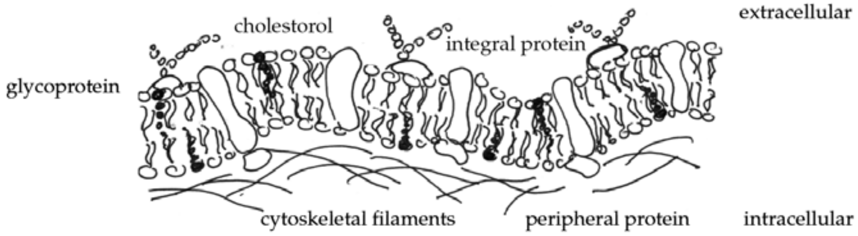

Figure 1.3. Cell membrane. Phospholipic bilayer with hydrophobic water-avoiding tails and hydrophilic water-loving heads.

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the lipid bilayer

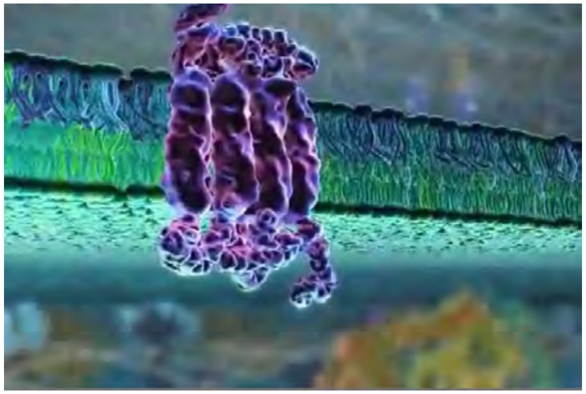

Figure 1.5.1. Lipid bilayer of the cell membrane. Characteristic arrangement of phospholipid molecules with hydrophilic polar head group being oriented towards the aqueous phase while the hydrophilic tails are oriented towards the non-polar inside.

the inner life of a cell, viel & lue, harvard [2006]

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the lipid bilayer - lipid rafts

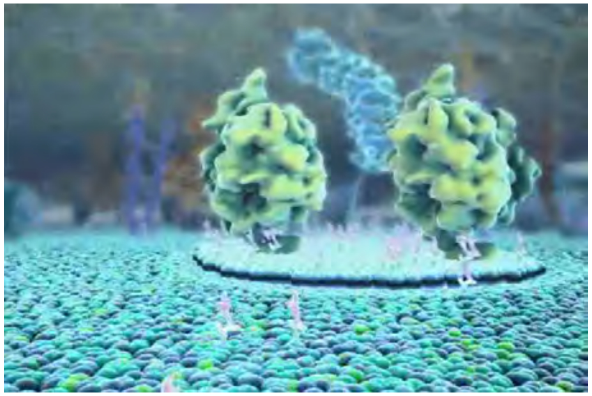

Figure 1.5.2. The lipid bilayer of the cell membrane is by no means static and homogeneous. Lipids are a class of molecules stacking together to form the membrane which can be understood as a sea on which things are floating. The rafts floating on this sea are called lipid rafts.

the inner life of a cell, viel & lue, harvard [2006]

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micropipette aspiration

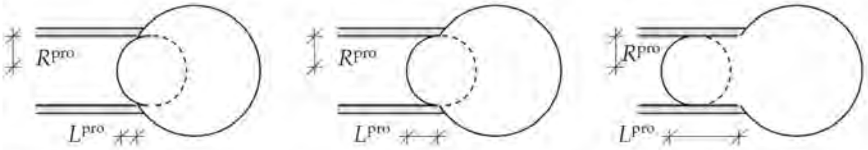


Figure 5.4: The three stages during mircopipette aspiration. The initial state with $L^{\rm pro}/R^{\rm pro} < 1$, left, the critical state with $L^{\rm pro}/R^{\rm pro} = 1$, middle, and the final state with $L^{\rm pro}/R^{\rm pro} > 1$, right.

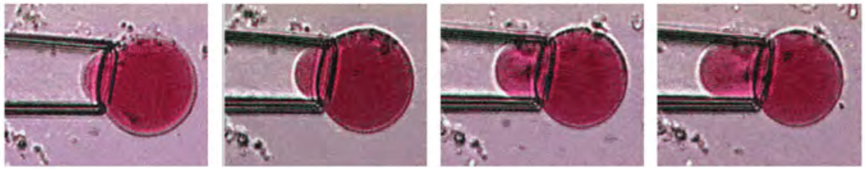


Figure 5.5: Experimental observation of different stages during micropipette aspriation adopted from http://newton.ex.ac.uk/research/biomedical/membranes.

surface tension

$$n = \sigma\, h \qquad \text{with} \qquad [\,n\,] = [\,\text{force / length}\,]$$

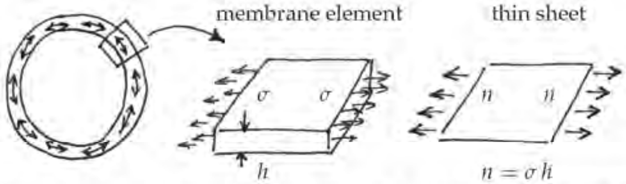
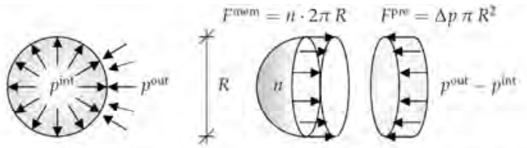


Figure 5.6: Liquid drop model. The internal fluid pressure is balanced by a thin elastic shell. The membrane element of thickness h is subjected to membrane stresses σ. Equivalently, the membrane can be represented as a thin sheet subjected to the surface tension n which results from the integration of the membrane stress over the thickness as $n = \sigma\, h$.

law of laplace



$$F^{\text{mem}} \doteq F^{\text{pre}}$$

where

$$F^{\text{mem}} = n\, C \qquad \text{with} \qquad C = 2\,\pi\, R$$

are the forces of the cell membrane acting along the circumference C and

$$F^{\text{pre}} = [\, p^{\text{int}} - p^{\text{out}} \,]\, A \qquad \text{with} \qquad A = \pi\, R^2$$

are the forces generated by the pressure difference across the cell wall acting on the surface area A. When combining these three equations, we obtain the law of Laplace

$$p^{\text{int}} - p^{\text{out}} = 2\, \frac{n}{R} \qquad \text{... Law of Laplace}$$

law of laplace

$$p^{\text{int}} - p^{\text{out}} = 2\, \frac{n}{R} \qquad \text{... Law of Laplace}$$

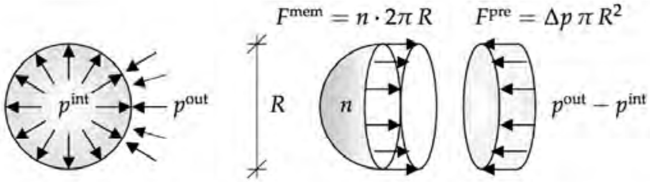


Figure 5.7: Law of Laplace. The membrane force $F^{\text{mem}} = n \cdot 2\pi\, R$ is the result of the surface tension n acting on the cell membrane along the circumference $C = 2\,\pi\, R$. It is in equilibrium with the forces F^{pre} resulting from the pressure difference Δp acting on the cell area $A = \pi\, R^2$.

law of laplace

$$\Delta p = 2\, n \left[\frac{1}{R^{\text{cell}}} - \frac{1}{R^{\text{pro}}} \right] \qquad \text{with} \qquad \Delta p = p^{\text{pip}} - p^{\text{out}}$$

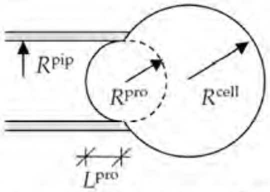


Figure 5.8: Kinematics of micropipette aspiration. For the limit state, at $L^{\text{pro}} / R^{\text{pro}} = 1$, the Law of Laplace can be used to determine the surface tension n.

law of laplace

$$p^{\text{pip}} + p^{\text{int}} - p^{\text{out}} = 2\,n\,\frac{1}{R^{\text{pip}}} \quad \ldots \quad \text{law of Laplace for the protrusion}$$

$$p^{\text{int}} - p^{\text{out}} = 2\,n\,\frac{1}{R^{\text{cell}}} \quad \ldots \quad \text{law of Laplace for the cell}$$

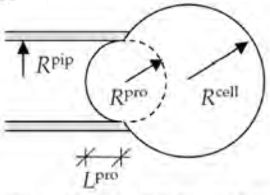
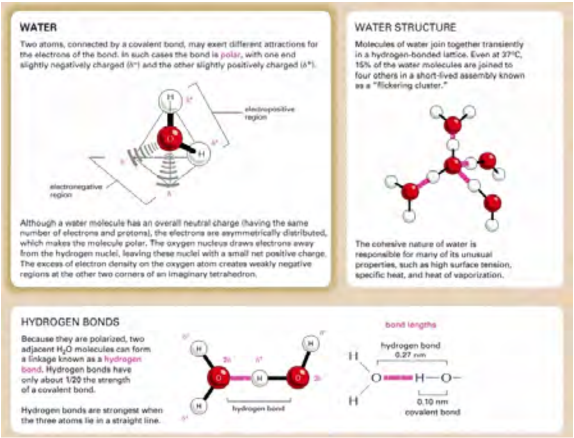


Figure 5.8: Kinematics of micropipette aspiration. For the limit state, at $L^{\text{pro}}/R^{\text{pro}} = 1$, the Law of Laplace can be used to determine the surface tension n.

$$p^{\text{pip}} = 2\,n\left[\frac{1}{R^{\text{pip}}} - \frac{1}{R^{\text{cell}}}\right]$$

---

the polar nature of water



WATER

Two atoms, connected by a covalent bond, may exert different attractions for the electrons of the bond. In such cases the bond is polar, with one end slightly negatively charged (δ^-) and the other slightly positively charged (δ^+).

Although a water molecule has an overall neutral charge (having the same number of electrons and protons), the electrons are asymmetrically distributed, which makes the molecule polar. The oxygen nucleus draws electrons away from the hydrogen nuclei, leaving these nuclei with a small net positive charge. The excess of electron density on the oxygen atom creates weakly negative regions at the other two corners of an imaginary tetrahedron.

WATER STRUCTURE

Molecules of water join together transiently in a hydrogen-bonded lattice. Even at 37°C, 15% of the water molecules are joined to four others in a short-lived assembly known as a "flickering cluster."

The cohesive nature of water is responsible for many of its unusual properties, such as high surface tension, specific heat, and heat of vaporization.

HYDROGEN BONDS

Because they are polarized, two adjacent H_2O molecules can form a linkage known as a hydrogen bond. Hydrogen bonds have only about 1/20 the strength of a covalent bond.

Hydrogen bonds are strongest when the three atoms lie in a straight line.

alberts, johnson, lewis, raff, roberts, walter [2002]

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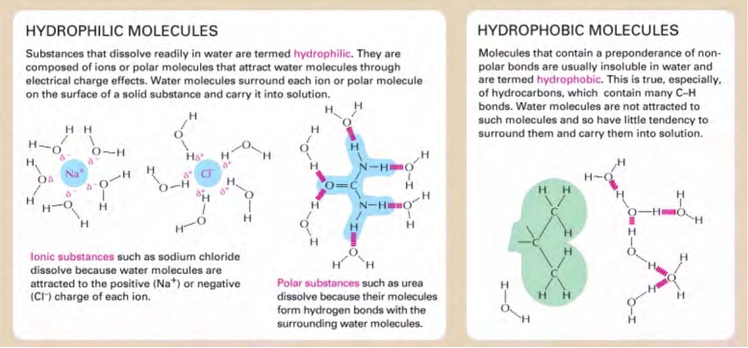
the polar nature of water



HYDROPHILIC MOLECULES

Substances that dissolve readily in water are termed hydrophilic. They are composed of ions or polar molecules that attract water molecules through electrical charge effects. Water molecules surround each ion or polar molecule on the surface of a solid substance and carry it into solution.

Ionic substances such as sodium chloride dissolve because water molecules are attracted to the positive (Na^+) or negative (Cl^-) charge of each ion.

Polar substances such as urea dissolve because their molecules form hydrogen bonds with the surrounding water molecules.

HYDROPHOBIC MOLECULES

Molecules that contain a preponderance of nonpolar bonds are usually insoluble in water and are termed hydrophobic. This is true, especially, of hydrocarbons, which contain many C–H bonds. Water molecules are not attracted to such molecules and so have little tendency to surround them and carry them into solution.

alberts, johnson, lewis, raff, roberts, walter [2002]

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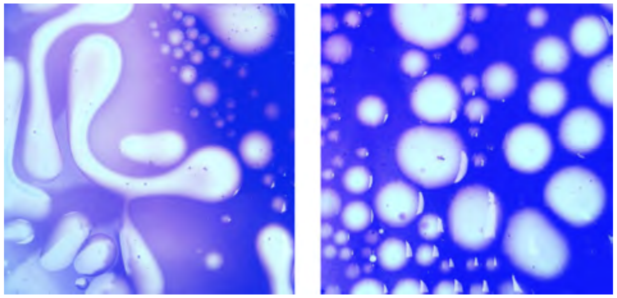
single layer - oil-water interface



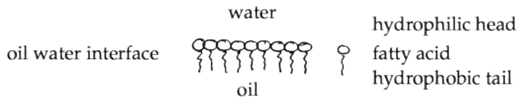


Figure 5.11: Oil water interface. Characteristic arrangement of fatty acid molecules with hydrophilic polar head group oriented towards the water phase and hydrophobic tail oriented towards the oil phase

double layer - water-oil-water interface

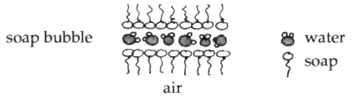


Figure 5.13: Lipid bilayer of soap bubbles - characteristic arrangement of soap molecules with a thin water layer being sandwiched between the hydrophilic polar head groups while the hydrophobic tails are oriented to the non-polar air

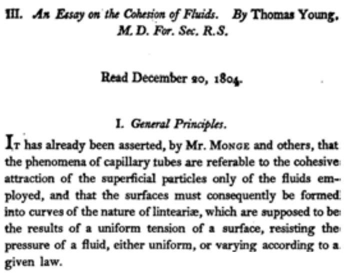
---

energy minimization

An interesting question to ask about soap bubbles is what is the radius r of a soap bubble that is blown up at a pressure Δp? Here $\Delta p = p^{\text{int}} - p^{\text{out}}$ would be the pressure difference between the inside and outside of the bubble. To answer this question, we consider spherical soap bubble with initial radius r, that has a surface of $A = 4\,\pi\,r^2$ and a volume of $V = \frac{4}{3}\pi\,r^3$. The inflation of the surface induces an internal energy W^{int} which is assumed to be proportional to the increase in membrane surface A. In the simplest case, $W^{\text{int}} = \gamma\,A$ where, for now, γ is introduces as a mere proportionality constant. Its unit is obvioulsy force per length and its physical interpretation will be discussed later. The external work W^{ext} is equal to the pressure difference Δp acting on the enclosed volume V, such that $W^{\text{ext}} = \Delta p\,V$. The total energy W of the bubble thus consists of the internal energy W^{int} and the external energy W^{ext}.

young [1805], laplace [1806]

energy minimization



III. *An Essay on the Cohesion of Fluids.* *By* Thomas Young, *M. D. For. Sec. R. S.*

Read December 20, 1804.

I. *General Principles.*

It has already been asserted, by Mr. Monge and others, that the phenomena of capillary tubes are referable to the cohesive attraction of the superficial particles only of the fluids employed, and that the surfaces must consequently be formed into curves of the nature of lintearæ, which are supposed to be the results of a uniform tension of a surface, resisting the pressure of a fluid, either uniform, or varying according to a given law.

$$W(r) = W^{\text{int}} - W^{\text{ext}}$$

$$W^{\text{int}} = \gamma\,A = \gamma\,4\,\pi\,r^2$$

$$W^{\text{ext}} = \Delta p\,V = \Delta p\,\tfrac{4}{3}\pi\,r^3$$

$$W(r) \to \min \qquad \delta W(r) \doteq 0$$

$$\delta W^{\text{int}} = \gamma\,8\,\pi\,r$$

$$\delta W^{\text{ext}} = \Delta p\,4\,\pi\,r^2$$

$$\gamma\,8\,\pi\,r - \Delta p\,4\,\pi\,r^2 \doteq 0 \qquad \Delta p = 2\,\gamma\,\frac{1}{r}$$

young [1805], laplace [1806]

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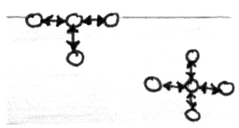
surface tension



air

air-water interface

water

no cohesive forces

cohesive forces from all sides

Figure 5.15: Air water interface - molecular interpretation of surface tension

Surface tension Surface tension is typically measured in force per length related to the units dynes per cm. Since 1 dyne = 10 mN, 1 dyne/cm = 1 mN/m. Alternatively, especially in thermodynamics, the notion surface energy is used instead. Surface energy is measured in ergs per length squared, where one eng, the force of one dyne exerted for a distance of one cm is equal to gram centimeter squared per second squared g cm^2/s^2 or, equivalently, 10^{-7} joules. The surface tension of water at room temperature is γ^{water}=72 dynes/cm, ethanol has a lower surface tension of γ^{ethanol}=22 dynes/cm and mercury has a surface tension as large as γ^{mercury}=465 dynes/cm.

the lipid bilayer

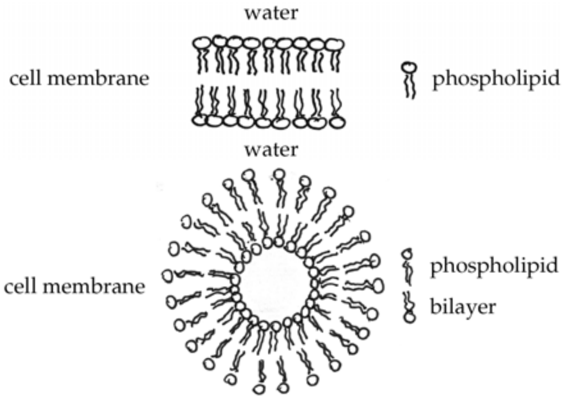


Figure 5.16. Lipid bilayer of the cell membrane. Characteristic arrangement of phospholipid molecules with hydrophilic polar head group being oriented towards the aqueous phase while the hydrophilic tails are oriented towards the non-polar inside.

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trusses, beams, walls, plates, membranes, shells

	dimension	geometry	loading	deformation	gov eqn
truss	1d straight	$w,h \ll l$	axial	tension	2^{nd} order
beam	1d straight	$w,h \ll l$	transverse	bending	4^{th} order
wall	2d flat	$h \ll w,l$	in plane	tension/shear	2^{nd} order
plate	2d flat	$h \ll w,l$	transverse	bending	4^{th} order
membrane	3d curved	$h \ll w,l$	in plane	tension/shear	2^{nd} order
shell	3d curved	$h \ll w,l$	transverse	bending	4^{th} order

Table 2.1: Classification of structural elements based on dimension, geometry and loading

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1d transverse deformation - bending

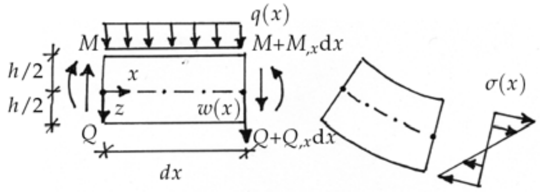


Figure 3.2: Transverse loading of one dimensional structure ∘ stresses σ vary linearly across the cross section

euler bernoulli beam theory

- normals remain straight (they do not bend)
- normals remain unstretched (they keep the same length)
- normals remain normal (they remain orthogonal to the beam axis)

---

tension vs bending - trusses vs beams

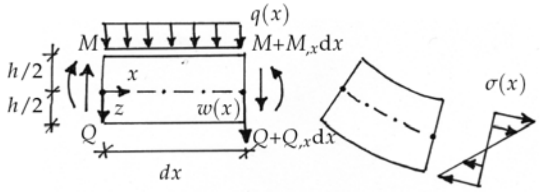


Figure 3.2: Transverse loading of one dimensional structure ∘ stresses σ vary linearly across the cross section

$$\text{overall deformation} = \boxed{\text{axial}} + \boxed{\text{transverse}}\ \text{deformation}$$

$$u^{\text{tot}}(x,z) = \boxed{u(x)}\ \boxed{-\, z\, w(x)_{,x}}$$

$$\varepsilon = u^{\text{tot}}_{,x} = \boxed{u_{,x}}\ \boxed{-\, z\, w_{,xx}}$$

- axial deformation $u(x)$
- transverse deformation, scaled rotation of beam axis $z\, w(x)_{,x}$

2d transverse deformation - bending

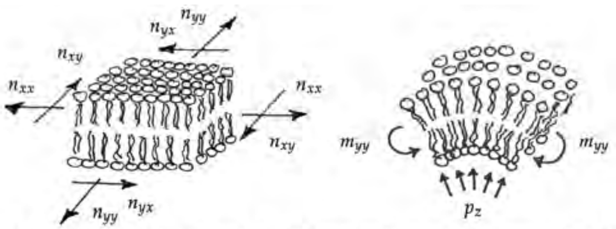


Figure 5.12: Infinitesimal element of the cell membrane subject to tension causing in plane deformation and shear (left) and bending causing out of plane deformation (right)

kirchhoff love shell theory

- normals remain straight (they do not bend)
- normals remain unstretched (they keep the same length)
- normals remain normal (they remain orthogonal to the beam axis)

---

tension vs bending - membranes vs shells

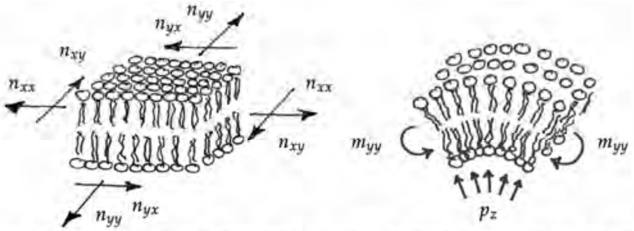


Figure 5.12: Infinitesimal element of the cell membrane subject to tension causing in plane deformation and shear (left) and bending causing out of plane deformation (right)

overall deformation = in plane + transverse deformation

$$\begin{aligned} u^{\text{tot}}(x,y,z) &= \boxed{u(x,y)}\,\boxed{-\,z\,w_{,x}} \\ v^{\text{tot}}(x,y,z) &= \boxed{v(x,y)}\,\boxed{-\,z\,w_{,y}} \\ w^{\text{tot}}(x,y,z) &= \boxed{w(x,y)} \end{aligned}$$

---

tension vs bending - membranes vs shells

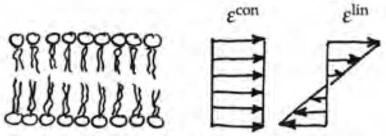


Figure 5.18: Von Kármán strains in cross section – constant terms ε^{con} related to in plain strains and linear terms ε^{lin} related to out of plane bending

overall strain = in plane (constant) + transverse (linear)

$$\begin{aligned} \varepsilon_{xx} &= \boxed{u_{,x} + \tfrac{1}{2}\,w_{,x}^2} - \boxed{z\,w_{,xx}} \\ \varepsilon_{yy} &= \boxed{v_{,y} + \tfrac{1}{2}\,w_{,y}^2} - \boxed{z\,w_{,yy}} \\ \varepsilon_{xy} &= \boxed{\tfrac{1}{2}\,[u_{,y} + v_{,x} + w_{,x}w_{,y}} - \boxed{2z\,w_{,xy}]} \end{aligned}$$

2d in-plane deformation - tension and shear

$$\begin{aligned} \varepsilon_{xx} &= u_{,x} + \tfrac{1}{2}\,w_{,x}^2 & \sigma_{xx} &= \frac{E}{1-\nu^2}\,[\,\varepsilon_{xx} + \nu\,\varepsilon_{yy}\,] \\ \varepsilon_{yy} &= v_{,y} + \tfrac{1}{2}\,w_{,y}^2 & \sigma_{yy} &= \frac{E}{1-\nu^2}\,[\,\varepsilon_{yy} + \nu\,\varepsilon_{xx}\,] \\ \varepsilon_{xy} &= \tfrac{1}{2}\,[\,u_{,y} + v_{,x}\,] + w_{,x}w_{,y} & \sigma_{xy} &= \frac{E}{1+\nu}\,\varepsilon_{xy} \end{aligned}$$

$$\begin{aligned} n_{xx} &= \frac{E\,h}{[1-\nu^2]}\,[\,\varepsilon_{xx} + \nu\,\varepsilon_{yy}\,] & \textstyle\sum f_x &\doteq 0 & n_{xx,x} + n_{xy,y} &= 0 \\ n_{yy} &= \frac{E\,h}{[1-\nu^2]}\,[\,\varepsilon_{yy} + \nu\,\varepsilon_{xx}\,] & \textstyle\sum f_y &\doteq 0 & n_{yx,x} + n_{yy,y} &= 0 \\ n_{xy} &= \frac{E\,h}{1+\nu}\,\varepsilon_{xy} & \textstyle\sum f_z &\doteq 0 \end{aligned}$$

$$n_{xx}\,w_{,xx} + 2\,n_{xy}\,w_{,xy} + n_{yy}\,w_{,yy} + p_z = 0$$

2d in-plane deformation - tension and shear

…this is just to show you how you could derive the force equilibrium

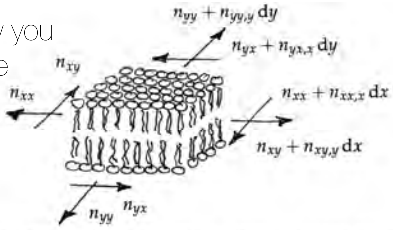


Figure 5.19: Infinitesimal element of the cell membrane with in plane tensile forces n_{xx} and n_{yy}

$$\begin{aligned}
\textstyle\sum f_x \doteq 0: \quad & -n_{xx}\mathrm{d}y + [n_{xx} + n_{xx,x}\mathrm{d}x]\mathrm{d}y - n_{yx}\mathrm{d}x + [n_{yx} + n_{yx,x}\mathrm{d}y]\mathrm{d}x = 0 \\
\textstyle\sum f_y \doteq 0: \quad & -n_{yy}\mathrm{d}x + [n_{yy} + n_{yy,y}\mathrm{d}y]\mathrm{d}x - n_{xy}\mathrm{d}y + [n_{xy} + n_{xy,y}\mathrm{d}x]\mathrm{d}y = 0 \\
\textstyle\sum f_z \doteq 0: \quad & -n_{xx}\mathrm{d}y\, w_{,x} + [n_{xx} + n_{xx,x}\mathrm{d}x]\mathrm{d}y[w_{,x} + w_{,xx}\mathrm{d}x] \\
& -n_{xy}\mathrm{d}y\, w_{,y} + [n_{xy} + n_{yx,x}\mathrm{d}x]\mathrm{d}y[w_{,y} + w_{,yx}\mathrm{d}x] \\
& -n_{yx}\mathrm{d}x\, w_{,x} + [n_{yx} + n_{xy,y}\mathrm{d}y]\mathrm{d}x[w_{,x} + w_{,xy}\mathrm{d}y] \\
& -n_{yy}\mathrm{d}x\, w_{,y} + [n_{yy} + n_{yy,y}\mathrm{d}y]\mathrm{d}x[w_{,y} + w_{,yy}\mathrm{d}y] + p_z\,\mathrm{d}x\mathrm{d}y = 0
\end{aligned}$$

---

membrane equations - in plane deformation

$$n_{xx}\, w_{,xx} + 2\, n_{xy}\, w_{,xy} + n_{yy}\, w_{,yy} + p_z = 0$$

now, let's look at some special cases…

- equilibiaxial tension (without shear) > extensional stiffness
- equilibiaxial tension (without shear) > area expansion modulus
- shear (without extension) > shear stiffness

---

membrane equations - in plane deformation

$$n_{xx}\, w_{,xx} + 2\, n_{xy}\, w_{,xy} + n_{yy}\, w_{,yy} + p_z = 0$$

special case of equibiaxial tension (without shear)

$$\sigma_{xx} = \sigma_{yy} = \sigma \qquad \sigma_{xy} = 0$$

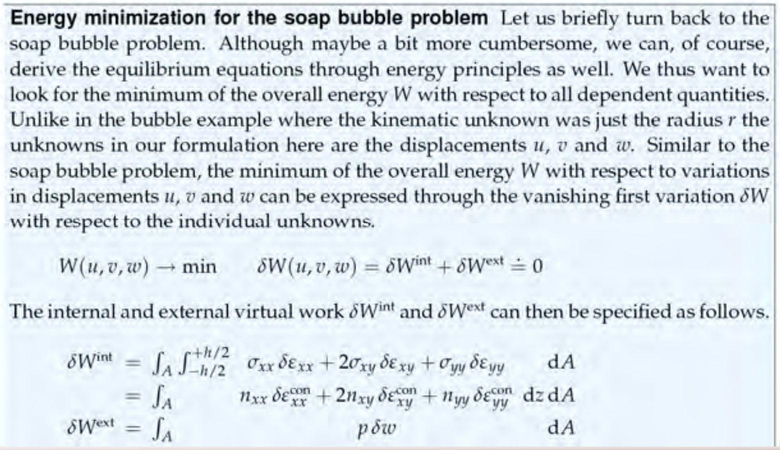
$$n_{xx} = n_{yy} = n \qquad n_{xy} = 0$$

$$n\,[\, w_{,xx} + w_{,yy}\,] + p_z = 0 \qquad w_{,xx} + w_{,yy} = \Delta w$$

$p_z = -n\,\Delta w$ with n ... surface tension

---

energy minimization



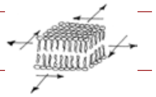
Energy minimization for the soap bubble problem Let us briefly turn back to the soap bubble problem. Although maybe a bit more cumbersome, we can, of course, derive the equilibrium equations through energy principles as well. We thus want to look for the minimum of the overall energy W with respect to all dependent quantities. Unlike in the bubble example where the kinematic unknown was just the radius r the unknowns in our formulation here are the displacements u, v and w. Similar to the soap bubble problem, the minimum of the overall energy W with respect to variations in displacements u, v and w can be expressed through the vanishing first variation δW with respect to the individual unknowns.

$$W(u,v,w) \rightarrow \min \qquad \delta W(u,v,w) = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} \doteq 0$$

The internal and external virtual work δW^{int} and δW^{ext} can then be specified as follows.

$$\begin{aligned}
\delta W^{\mathrm{int}} &= \int_A \int_{-h/2}^{+h/2} \sigma_{xx}\,\delta\varepsilon_{xx} + 2\sigma_{xy}\,\delta\varepsilon_{xy} + \sigma_{yy}\,\delta\varepsilon_{yy} \quad \mathrm{d}A \\
&= \int_A n_{xx}\,\delta\varepsilon^{\mathrm{con}}_{xx} + 2n_{xy}\,\delta\varepsilon^{\mathrm{con}}_{xy} + n_{yy}\,\delta\varepsilon^{\mathrm{con}}_{yy} \quad \mathrm{d}z\,\mathrm{d}A \\
\delta W^{\mathrm{ext}} &= \int_A p\,\delta w \quad \mathrm{d}A
\end{aligned}$$

energy minimization

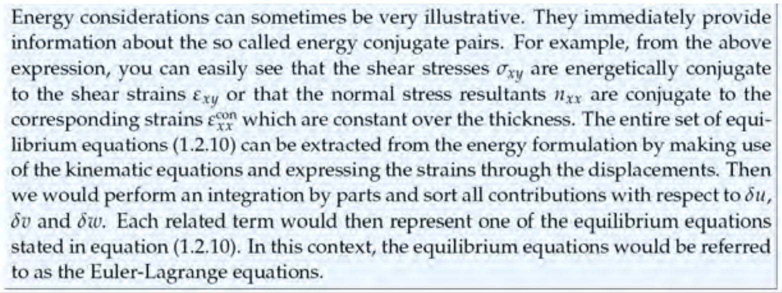


compare to young and laplace solution

$$W(r) = W^{\text{int}} - W^{\text{ext}}$$

$$\begin{aligned} W^{\text{int}} &= \gamma A = \gamma\, 4\,\pi\, r^2 \\ W^{\text{ext}} &= \Delta p\, V = \Delta p\, \tfrac{4}{3}\pi\, r^3 \end{aligned}$$

$$\Delta p = 2\,\gamma\,\frac{1}{r}$$



Energy considerations can sometimes be very illustrative. They immediately provide information about the so called energy conjugate pairs. For example, from the above expression, you can easily see that the shear stresses σ_{xy} are energetically conjugate to the shear strains ε_{xy} or that the normal stress resultants n_{xx} are conjugate to the corresponding strains $\varepsilon_{xx}^{\text{con}}$ which are constant over the thickness. The entire set of equilibrium equations (1.2.10) can be extracted from the energy formulation by making use of the kinematic equations and expressing the strains through the displacements. Then we would perform an integration by parts and sort all contributions with respect to δu, δv and δw. Each related term would then represent one of the equilibrium equations stated in equation (1.2.10). In this context, the equilibrium equations would be referred to as the Euler-Lagrange equations.